CHAPTER 2

Reasoning in Geometry

Content Summary

One major purpose of any geometry course is to improve the ability of students to reason logically. Chapter 2 focuses on two basic kinds of reasoning: inductive and deductive. Students use inductive reasoning to identify visual and numerical patterns and to make predictions based on these patterns. Then students are introduced to the use of deductive reasoning to explain why these patterns are true. Students explore the relationships between the measures of angles formed by intersecting and parallel lines, make conjectures about these relationships, and learn how to use logical arguments to explain why these conjectures are true.

Inductive Reasoning

Every time you freeze your water bottle, the water expands. So you learn quickly not to put too much water in the bottle, to avoid having the top pop off or the bottle break. Reasoning in which you draw conclusions from experience is inductive. Mathematicians use inductive reasoning to guess at what might be true. For example, if you add up positive odd numbers starting at 1, you get a pattern.

1 + 3 = 4
1 + 3 + 5 = 9
1 + 3 + 5 + 7 = 16

The sums seem to be square numbers: 4 = 2 · 2, 9 = 3 · 3, 16 = 4 · 4. Even if you “add up” only the first odd number, 1, the sum is a square: 1 · 1. You might conclude that the sum of any number of positive consecutive odd numbers, beginning with 1, is a square number.

But inductive reasoning is not foolproof. It may be that one time you’ll put the water bottle in the freezer and the water won’t expand. (Perhaps the freezer isn’t working, or the water is flavored in some way.) So mathematicians are not satisfied with inductive reasoning. Inductive reasoning leads only to guesses, or conjectures. A conjecture becomes a mathematical fact, or theorem, only if someone shows that it’s the conclusion of deductive reasoning.

Deductive Reasoning

Deductive reasoning, also called proof, is reasoning from proven facts using logically valid steps to arrive at a conclusion. A proof can serve many purposes. Mathematicians often use proof to verify that a conjecture is true for all cases, not just for those they examined, or to convince others. Proofs often help answer the question: Why? The use of proof to explain why is a natural extension for students at this point in the course and helps to deepen their understanding. This chapter stresses this illumination purpose of proof.

If you took a geometry course, you may have encountered proofs displayed in two columns: a column of statements and a column of reasons, with each statement justified by a reason. However, most students are overwhelmed by this approach. They find the format difficult to follow and miss the big picture. In Discovering Geometry, two-column proofs come in Chapter 13 with the study of geometry as a mathematical system, at which point students are at the appropriate developmental...
Chapter 2 • Reasoning in Geometry (continued)

level. For now, students are encouraged to use informal deductive arguments written in a paragraph form. In Chapter 4, they will be introduced to other formats for presenting proofs.

Reasoning Strategies
The most difficult part of the process for writing a deductive argument is to determine the underlying logic of the argument and what information to include. Starting in Chapter 2 and continuing throughout the book, students are taught reasoning strategies, ways of thinking that help to construct a deductive argument. You may wish to discuss these ways of thinking with your student. The first three of these reasoning strategies are presented in this chapter, and one more strategy is introduced in each subsequent chapter as indicated.

- Draw a labeled diagram and mark what you know.
- Represent a situation algebraically.
- Apply previous conjectures and definitions.
- Break a problem into parts (Chapter 3)
- Add an auxiliary line (Chapter 4)
- Think backward (Chapter 5)

Summary Problem
Draw two intersecting lines. What do you notice about the vertical angles? Can you explain any patterns you see?

Questions you might ask in your role as student to your student:

- What does the term *vertical angles* mean?
- Which pairs of angles are vertical angles?
- What conjecture might you state about the measures of vertical angles?
- Does the conjecture you’re making apply to all pairs of intersecting lines that you’ve tried, or can you find a counterexample?
- Do you think your conjecture holds for all pairs of intersecting lines?
- How would you show that it’s true for all pairs?

Sample Answers
When two lines intersect, they form four distinct angles. Two nonadjacent angles formed by the intersecting lines are vertical angles. Two intersecting lines form two pairs of vertical angles. The angles in each pair have equal measures. Explaining why could involve talking about straight lines and the various pairs of adjacent angles, or about a rotation around the vertex. You can write a deductive argument to show that the Vertical Angles Conjecture follows logically from the Linear Pair Conjecture, as shown on page 124.
Chapter 2 • Review Exercises

1. (Lesson 2.1) Use the rule provided to generate the next five terms of the sequence.
   \[ 2, 5, 9, 14, \ldots, \frac{n(n + 3)}{2}, \ldots \]

2. (Lesson 2.2, 2.3, 2.4) Find the rule for the number of circles in the \( n \)th figure, and use that to find the number of circles in the 20th figure. Are you using inductive or deductive reasoning to answer this question?

   

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<thead>
<tr>
<th>Figure number</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>( n )</th>
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<tbody>
<tr>
<td>Number of circles</td>
<td>1</td>
<td>6</td>
<td>11</td>
<td>16</td>
<td>( \ldots )</td>
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3. (Lessons 2.5, 2.6) Find the measure of each lettered angle. Is line \( l \) parallel to \( m \)? Explain your reasoning.

4. (Lesson 2.6) Lines \( l \) and \( m \) are parallel to each other. Find \( x \).
Therefore, the rule is $5n - 4$. To find out how many circles there are in the 20th figure, substitute 20 for $n$ in the rule.

$5(20) - 4 = 96$

3. $a = 55^\circ$ by the Vertical Angles Conjecture.
   $b = 126^\circ$ by the Vertical Angles Conjecture.
   $c = 54^\circ$ by the Linear Pair Conjecture.
   If lines $l$ and $m$ were parallel, then $c$ would equal $55^\circ$ by the Corresponding Angles Conjecture. However, $c = 54^\circ$ as seen above, so $l$ and $m$ are not parallel.

4. $40 + 3x + 4x = 180$ Corresponding Angles Conjecture and Linear Pair Conjecture.
   
   $40 + 7x = 180$ Combine like terms.
   
   $7x = 140$ Subtract 40 from both sides.
   
   $x = 20$ Divide both sides by 7.

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1. 20, 27, 35, 44, 54

The first four terms come from substituting 1, 2, 3, and 4 for $n$ in the given rule. Substitute 5, 6, 7, 8, and 9 for $n$ to find the next five terms. See bottom of the page.

2. $5n - 4$ (or $5(n - 1) + 1$, which is equivalent); 96; inductive reasoning

First, look for the difference between the terms. In this case we add five to each term to get the next term.

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Because the difference between consecutive terms is always five, the rule is $5n + "$something." Let $c$ stand for the unknown "something," so the rule is $5n + c$.

To find $c$, replace $n$ in the rule with one term number. For example, try $n = 3$ and set the expression equal to 11, the number of circles in the 3rd figure.

$5(3) + c = 11$

$15 + c = 11$

$c = -4$

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1. 

<table>
<thead>
<tr>
<th>$n$</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
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<tbody>
<tr>
<td>$n(n + 3)$</td>
<td>$\frac{5(5 + 3)}{2} = 20$</td>
<td>$\frac{6(6 + 3)}{2} = 27$</td>
<td>$\frac{7(7 + 3)}{2} = 35$</td>
<td>$\frac{8(8 + 3)}{2} = 44$</td>
<td>$\frac{9(9 + 3)}{2} = 54$</td>
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