

6.1

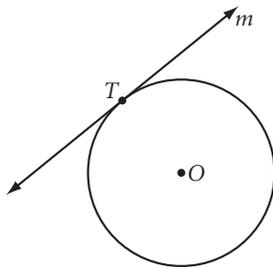
Tangent Properties

In this lesson you will

- Review terms associated with **circles**
- Discover how a tangent to a circle and the radius to the **point of tangency** are related
- Make a conjecture about **tangent segments** to a circle from a point outside the circle

You have already learned some terms related to circles. Review them by matching the terms and figures on page 310 of your book.

The **tangent** to a circle is a line in the plane of the circle that intersects the circle in exactly one point. The point where the tangent touches the circle is the **point of tangency**.



Line m is a **tangent** to circle O .
Point T is the **point of tangency**.

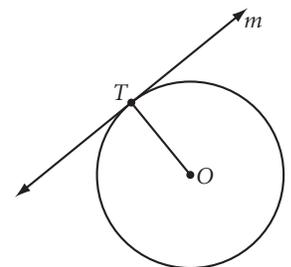
Investigation 1: Going Off on a Tangent

Construct a large circle. Label the center O .

Use a straightedge to draw a line that appears to touch the circle at one point. Label the point T and draw radius \overline{OT} .

Measure the angles formed by the radius and the tangent line.
How do the angles compare?

Will you get the same result with a different tangent? Draw another tangent and see. Use your findings to complete this conjecture.



Tangent Conjecture A tangent to a circle is _____ to the radius drawn to the point of tangency.

C-53

Investigation 2: Tangent Segments

Follow Steps 1–3 in your book. How do the lengths of the tangent segments compare?

Will you get the same results if you start with a different point outside the circle? Choose another point and see. Use your findings to complete this conjecture.

Tangent Segments Conjecture Tangent segments to a circle from a point outside the circle are _____.

C-54

(continued)

Lesson 6.1 • Tangent Properties (continued)

Read the information about minor and major arcs on page 313 of your book, and study the example. Then read about **internally tangent circles** and **externally tangent circles**.

Now read the example below.

EXAMPLE

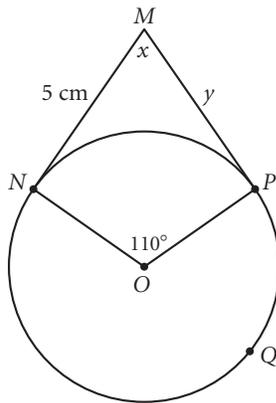
- a. \overline{MN} and \overline{MP} are tangents to circle O .

$$x = \underline{\hspace{2cm}}$$

$$y = \underline{\hspace{2cm}}$$

$$m\widehat{NP} = \underline{\hspace{2cm}}$$

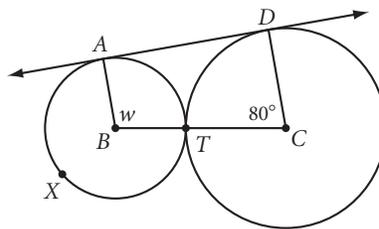
$$m\widehat{PQN} = \underline{\hspace{2cm}}$$



- b. \overline{AD} is tangent to both circle B and circle C .

$$w = \underline{\hspace{2cm}}$$

$$m\widehat{AXT} = \underline{\hspace{2cm}}$$



- c. Refer to the figure for part a. What type of quadrilateral is $MNOP$?

► Solution

- a. By the Tangent Conjecture, $m\angle MNO = 90^\circ$ and $m\angle MPO = 90^\circ$. By the Quadrilateral Sum Conjecture, $90^\circ + 90^\circ + 110^\circ + x = 360^\circ$, so $x = 70^\circ$. By the Tangent Segments Conjecture, $\overline{MN} \cong \overline{MP}$, so $y = 5$ cm.

Because the measure of a minor arc equals the measure of its central angle, $m\widehat{NP} = 110^\circ$. Subtract this measure from 360° to find the measure of the major arc, giving $m\widehat{PQN} = 250^\circ$.

- b. By the Tangent Conjecture, $m\angle BAD = 90^\circ$ and $m\angle CDA = 90^\circ$. By the Quadrilateral Sum Conjecture, $90^\circ + 90^\circ + 80^\circ + w = 360^\circ$, so $w = 100^\circ$.

The measure of minor arc AT equals the measure of the central angle $\angle ABT$, so $m\widehat{AT} = 100^\circ$. Subtract this measure from 360° to find the measure of its major arc: $m\widehat{AXT} = 360^\circ - 100^\circ = 260^\circ$.

- c. $MNOP$ is a kite because $\overline{MN} \cong \overline{MP}$ and $\overline{ON} \cong \overline{OP}$.

6.2

Chord Properties

In this lesson you will

- Define **central angle** and **inscribed angle**
- Investigate properties of **chords** of a circle

Investigation 1: Defining Angles in a Circle

In your book, look at the examples and non-examples of central angles. What do the central angles have in common? What characteristics do the central angles have that the other angles do not have? Use your observations to complete this definition of a central angle.

A **central angle** is an angle whose vertex is _____.

Now look at the examples and non-examples of inscribed angles. Notice that each inscribed angle has its vertex on the circle. However, so does $\angle VWX$, which is *not* an inscribed angle. What makes the inscribed angles different from $\angle VWX$? Use your findings to complete this definition.

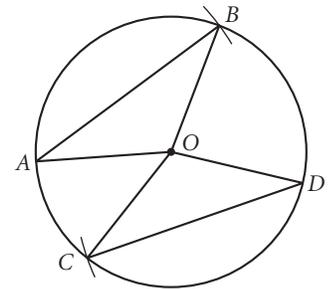
An **inscribed angle** is an angle whose vertex is _____ and whose sides _____.

Investigation 2: Chords and Their Central Angles

Construct a large circle and label its center O . Use your compass to construct two congruent chords. Label the chords \overline{AB} and \overline{CD} . Construct radii \overline{OA} , \overline{OB} , \overline{OC} , and \overline{OD} .

Use a protractor to measure $\angle BOA$ and $\angle COD$. How do the measures compare?

See if you get the same results if you start with a different circle. Then complete this conjecture.



Chord Central Angles Conjecture If two chords in a circle are congruent, then they determine two central angles that are _____.

C-55

You learned in Lesson 6.1 that a minor arc has the same measure as its central angle. If chords \overline{AB} and \overline{CD} are congruent and central angles $\angle BOA$ and $\angle COD$ are congruent, what can you conclude about their intercepted arcs, \widehat{AB} and \widehat{CD} ?

You can fold your paper to confirm your answer. Now complete the conjecture.

Chord Arcs Conjecture If two chords in a circle are congruent, then their _____ are congruent.

C-56

(continued)

Lesson 6.2 • Chord Properties (continued)

Investigation 3: Chords and the Center of the Circle

Follow Step 1 in your book. The perpendicular to each chord divides the chord into two segments. How do the lengths of the segments compare? Use your findings to complete this conjecture.

Perpendicular to a Chord Conjecture The perpendicular from the center of a circle to a chord is the _____ of the chord. **C-57**

Now use your compass to compare the distances (measured along the perpendicular) from the center to the chords. How do the distances compare?

Draw a circle of a different size and follow Step 1 in your book for the new circle. Compare the distances from the center of the circle to each chord. Are your findings the same as they were for the first circle? Complete the conjecture.

Chord Distance to Center Conjecture Two congruent chords in a circle are _____ from the center of the circle. **C-58**

Investigation 4: Perpendicular Bisector of a Chord

Construct a large circle and mark the center. Draw two nonparallel chords that are *not* diameters. Construct the perpendicular bisector of each chord and extend the bisectors until they intersect. Where is the point of intersection?

Construct another circle that is larger or smaller than the first circle you drew. Draw two chords and construct their perpendicular bisectors. Where do the perpendicular bisectors intersect? Complete this conjecture.

Perpendicular Bisector of a Chord Conjecture The perpendicular bisector of a chord passes through the _____ of the circle. **C-59**

Now you can find the center of any circle and the vertex of the central angle of any arc. All you have to do is construct the perpendicular bisectors of nonparallel chords.

6.3

Arcs and Angles

In this lesson you will

- Make conjectures about **inscribed angles** in a circle
- Investigate relationships among the angles in a **cyclic quadrilateral**
- Compare the arcs formed when two parallel lines intersect a circle

In this lesson you'll discover properties of arcs and the angles associated with them.

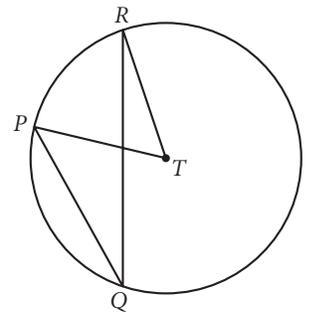
Investigation 1: Inscribed Angle Properties

Refer to the diagram of circle O in your book. Notice that $\angle COR$ is a *central angle*, while $\angle CAR$ is an *inscribed angle*. Both angles intercept \widehat{CR} .

Recall that the measure of a minor arc is equal to the measure of its central angle. Find $m\widehat{CR}$ by measuring $\angle COR$. Then find the measure of $\angle CAR$. How does the measure of the inscribed angle compare with the measure of its intercepted arc?

Construct your own circle T . Draw an inscribed angle PQR and its corresponding central angle PTR . An example is shown at right.

What is the measure of \widehat{PR} ? What is the measure of $\angle PQR$? How do the measures compare? Use your findings from this investigation to complete the conjecture.



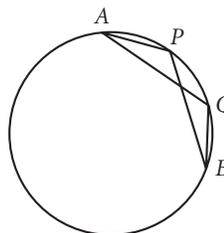
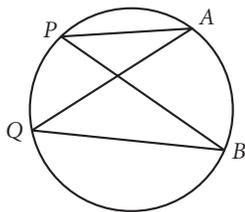
Inscribed Angle Conjecture The measure of an angle inscribed in a circle is _____ the measure of its intercepted arc.

C-60

Investigation 2: Inscribed Angles Intercepting the Same Arc

Follow Steps 1 and 2 in your book. Note that selecting P and Q on the major arc creates two angles, $\angle APB$ and $\angle AQB$, that intercept the same *minor arc*.

Draw another circle. Follow Step 4 in your book. Note that selecting P and Q on the minor arc creates two angles, $\angle APB$ and $\angle AQB$, that intercept the same *major arc*.



In each case, how do the measures of the two inscribed angles compare? Use your findings to complete the conjecture.

Inscribed Angles Intercepting Arcs Conjecture Inscribed angles that intercept the same arc are _____.

C-61

(continued)

Lesson 6.3 • Arcs and Angles (continued)

Investigation 3: Angles Inscribed in a Semicircle

Follow Steps 1 and 2 in your book to construct and measure three angles inscribed in the same semicircle. All the angles should have the same measure. What is the measure? Complete this conjecture.

Angles Inscribed in a Semicircle Conjecture Angles inscribed in a semicircle are _____.

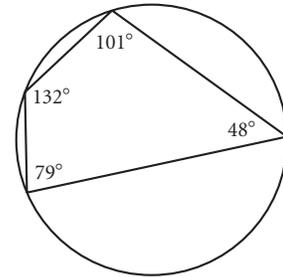
C-62

Investigation 4: Cyclic Quadrilaterals

A quadrilateral inscribed in a circle is called a **cyclic quadrilateral**.

Construct a large circle. Choose four points on the circle and connect them to form a cyclic quadrilateral. Label each angle with its measure. In another circle, construct a second cyclic quadrilateral and label its angles with their measures. At right is one more example.

In each cyclic quadrilateral, find sums of pairs of consecutive angles and sums of pairs of opposite angles. What do you discover? Complete this conjecture.



Cyclic Quadrilateral Conjecture The _____ angles of a cyclic quadrilateral are _____.

C-63

Investigation 5: Arcs by Parallel Lines

Read the definition of **secant** and follow Steps 1–3 in your book. Then complete the conjecture.

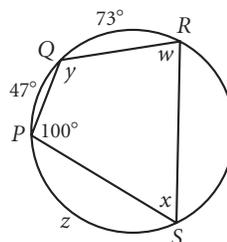
Parallel Lines Intercepted Arcs Conjecture Parallel lines intercept _____ arcs on a circle.

C-64

Read the example below.

EXAMPLE

Find each lettered measure.



► Solution

By the Cyclic Quadrilateral Conjecture, $w + 100^\circ = 180^\circ$, so $w = 80^\circ$. $\angle PSR$ intercepts \widehat{PR} . $m\widehat{PR} = 47^\circ + 73^\circ = 120^\circ$, so by the Inscribed Angle Conjecture, $x = \frac{1}{2}(120^\circ) = 60^\circ$.

By the Cyclic Quadrilateral Conjecture, $x + y = 180^\circ$. Substituting 60° for x and solving the equation gives $y = 120^\circ$. By the Inscribed Angle Conjecture, $w = \frac{1}{2}(47^\circ + z)$. Substituting 80° for w and solving the equation gives $z = 113^\circ$.

6.4

Proving Circle Conjectures

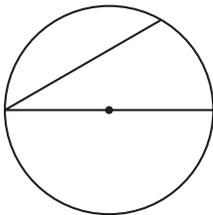
In this lesson you will

- Prove the **Inscribed Angle Conjecture**
- Prove other conjectures related to circles

In Lesson 6.3, you discovered the Inscribed Angle Conjecture.

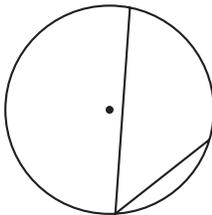
The measure of an inscribed angle in a circle equals half the measure of its intercepted arc.

In this lesson you will prove this conjecture. First, note that there are three ways an inscribed angle can relate to the circle's center.



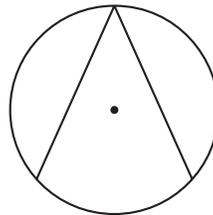
Case 1

The circle's center is on the angle.



Case 2

The center is outside the angle.



Case 3

The center is inside the angle.

These are the only possibilities, so if you can prove the conjecture is true for each case, you will have proved the conjecture is always true.

Case 1: The circle's center is on the inscribed angle.

Look at the drawing for Case 1 on page 330 of your book. The dashed radius \overline{OA} has been added to form $\triangle AOB$. To prove the conjecture is true for this case, you need to show that $\angle ABC = \frac{1}{2}m\widehat{AC}$ (that is, $x = \frac{1}{2}m\widehat{AC}$).

Notice the following things:

- $\angle AOC$ is the central angle for \widehat{AC} . (How are z and $m\widehat{AC}$ related?)
- $\angle AOC$ is an exterior angle for $\triangle AOB$. (How are x , y , and z related?)
- $\triangle AOB$ is isosceles. (How are x and y related?)

Combine these observations to write a flowchart proof for Case 1. Then, compare your proof with the one in your book.

Case 2: The circle's center is outside the inscribed angle.

You can use Case 1 to help you prove Case 2. Look at the drawing for Case 2 on page 331 of your book. A dashed diameter, \overline{DB} , has been added. To prove the conjecture is true for this case, you must show that $m\angle ABC = \frac{1}{2}m\widehat{AC}$ (that is, $x = \frac{q}{2}$).

Think about the following things:

- How are x , y , and z related?
- How are p , q , and $m\widehat{DC}$ related?
- Using Case 1, how is y related to p ? How is z related to $p + q$?

(continued)

Lesson 6.4 • Proving Circle Conjectures (continued)

Combine these observations to write a flowchart proof for Case 2. Then compare your proof with the one in your book.

Case 3: The circle's center is inside the inscribed angle.

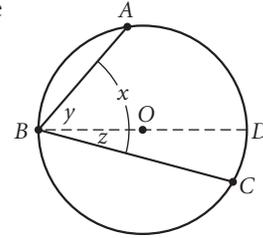
Use the given information and the marked figure in Example A below to write a flowchart proof for this case. You may want to use your flowchart from Case 2 as a guide for this proof. Write your proof before you look at the solution to Example A. Then compare your proof with the one given.

EXAMPLE A

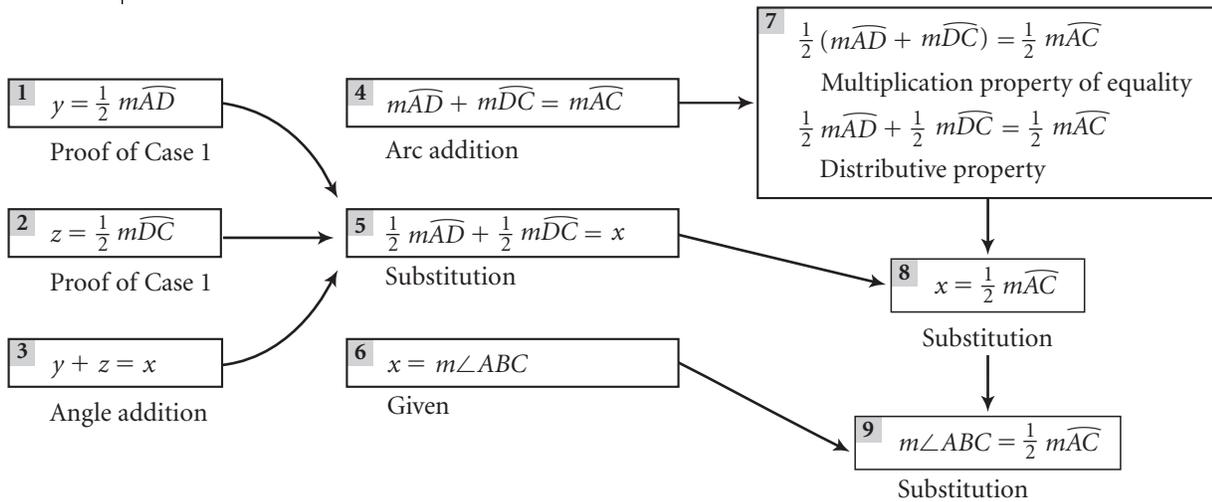
Prove Case 3 of the Inscribed Angle Conjecture: The measure of an inscribed angle in a circle equals half the measure of its intercepted arc when the center of the circle is inside the angle.

Given: Circle O with inscribed angle ABC whose sides lie on either side of diameter BD

Show: $m\angle ABC = \frac{1}{2}m\widehat{AC}$, or $x = \frac{1}{2}m\widehat{AC}$



► Solution

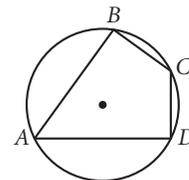


EXAMPLE B

Write a paragraph proof or a flowchart proof of the conjecture: If a kite is inscribed in a circle, then the nonvertex angles are right angles.

Given: Circle with inscribed kite $ABCD$ where $\overline{AB} \cong \overline{AD}$ and $\overline{BC} \cong \overline{CD}$

Show: $\angle B$ and $\angle D$ are right angles



► Solution

By the Kite Angles Conjecture, $\angle B \cong \angle D$. By the Cyclic Quadrilateral Conjecture, $\angle B$ and $\angle D$ are supplementary. Because $\angle B$ and $\angle D$ are both equal and supplementary, they must have measures of 90° . Therefore, $\angle B$ and $\angle D$ are right angles.

6.5

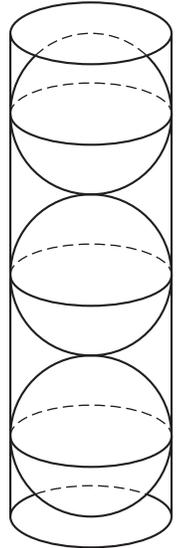
The Circumference/Diameter Ratio

In this lesson you will

- Discover the relationship between the **diameter** and the **circumference** of a circle

The distance around a polygon is called the perimeter. The distance around a circle is called the **circumference**.

Think of a can of three tennis balls. Which do you think is greater, the height of the can or the circumference of the can? The height is about three tennis-ball diameters, which is about the same as three can diameters. If you have a tennis-ball can, wrap a string around the can to measure the circumference, and then compare the measurement with the height. You should find that the circumference of the can is slightly greater than the height. That is, the circumference is a little more than three diameters. In the next investigation you will discover the relationship between the diameter and circumference of any circle.



Investigation: A Taste of Pi

For this investigation you will need several round objects, along with either a metric measuring tape or a string and a meterstick.

Wrap the measuring tape or string around each object to measure its circumference. Then measure the diameter of each object. Make all measurements to the nearest millimeter (tenth of a centimeter). The table below includes some sample measurements. Copy the table and add your findings to it.

Object	Circumference (C)	Diameter (d)	Ratio $\frac{C}{d}$
Quarter	7.8 cm	2.5 cm	
Compact disc	37.7 cm	12.0 cm	
Mug	25.9 cm	8.2 cm	
Small plate	47.4 cm	15.1 cm	

Calculate the ratio $\frac{C}{d}$ for each object. You should find that, in each case, the ratio is a little more than 3. In fact, the ratio $\frac{C}{d}$ is exactly the same number for every circle. This constant ratio is denoted by the Greek letter π (pi). So, $\frac{C}{d} = \pi$.

If you solve this equation for C, you get the formula for the circumference of a circle in terms of its diameter. Because the diameter of a circle is twice the radius, you can also write a formula for the circumference in terms of the radius.

Circumference Conjecture If C is the circumference and d is the diameter of a circle, then there is a number π such that $C =$ _____.
If $d = 2r$ where r is the radius, then $C =$ _____.

C-65

(continued)

Lesson 6.5 • The Circumference/Diameter Ratio (continued)

The number π is an *irrational number*, so its decimal form never ends and its digits follow no pattern. Your calculator probably gives an approximation of π to eight or nine decimal places. If you don't have access to a calculator, you can use the value 3.14 as an approximation for π . If you are asked to give an exact answer, state your answer in terms of π . The examples in your book show you how you can apply the Circumference Conjecture. Read the examples carefully. Here are two more examples.

EXAMPLE A | A circle has a radius of 6.5 meters. What is its circumference? Use a calculator and give your answer to the nearest 0.1 meter.

► **Solution** | $C = 2\pi r$ The formula for circumference.
 $C = 2\pi(6.5)$ Substitute 6.5 for r .
 $C = 13\pi \approx 40.8$ m Evaluate and round to the nearest 0.1 m.

EXAMPLE B | To find your hat size, you measure the circumference of your head in inches, and then use the circumference formula to find the diameter to the nearest eighth of an inch. The circumference of Tameka's head is about $23\frac{1}{2}$ in. What is her hat size?

► **Solution** | $C = \pi d$ The formula for circumference.
 $23\frac{1}{2} = \pi d$ Substitute $23\frac{1}{2}$ for C .
 $7\frac{1}{2} \approx d$ Divide both sides by π and round to the nearest eighth of an inch.
Tameka's hat size is $7\frac{1}{2}$.

6.6

Around the World

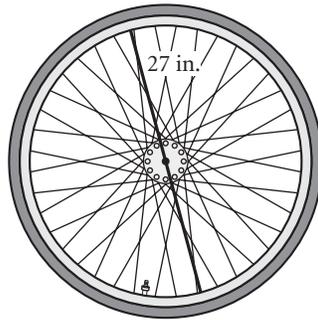
In this lesson you will

- Solve application problems involving radius, diameter, and circumference

Now that you have learned about π and the formula for circumference, you can solve many real-world problems involving circles. The example in your book shows the solution to a problem related to the novel *Around the World in Eighty Days*, by Jules Verne. Read this example carefully. Then read the examples below.

EXAMPLE A

The wheel on Devin's unicycle has a diameter of 27 inches.



- How far will the unicycle travel in 100 revolutions of the wheel? Give your answer to the nearest whole foot.
- Devin rides his unicycle 2 miles to school each day. About how many revolutions does the wheel make during his trip? (5280 feet = 1 mile)

► Solution

- In one revolution, the wheel covers a distance equal to its circumference.

$$C = \pi d \quad \text{The formula for circumference.}$$

$$C = \pi(27) \quad \text{Substitute 27 for } d.$$

The unicycle travels 27π inches in one revolution. So, in 100 revolutions, it covers $(100) \cdot 27\pi \approx 8482$ inches, or about 707 feet.

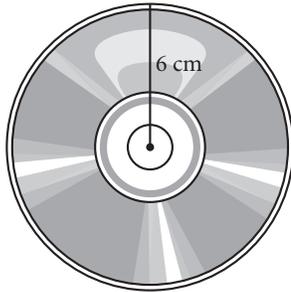
- Two miles is 10,560 feet, or 126,720 inches. The wheel makes one revolution every 27π inches, so it makes $126,720 \div 27\pi$, or about 1494 revolutions during Devin's trip to school.

(continued)

Lesson 6.6 • Around the World (continued)

EXAMPLE B

An audio CD spins at a rate of 200 rotations per minute. If the radius of a CD is 6 cm, how far does a point on the outer edge travel during the playing of a 57-minute CD?



► Solution

The point makes 200 rotations in 1 minute. So, in 57 minutes, it will make $200 \cdot 57$, or 11,400 rotations.

The distance the point travels in one rotation is equal to the circumference of the CD. Use the circumference formula to find this distance.

$$C = 2\pi r \quad \text{The formula for circumference.}$$

$$C = 2\pi(6) \quad \text{Substitute 6 for } r.$$

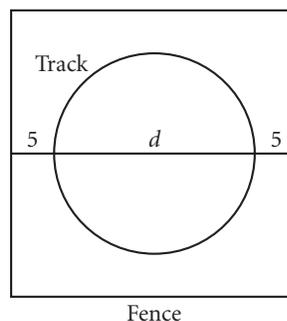
The point travels 12π cm in one rotation. So, in 11,400 rotations, it will travel $11,400 \cdot 12\pi$, or 429,770 cm. This is equal to about 4.3 kilometers.

EXAMPLE C

The city of Rutledge wants to build a circular walking track that is a quarter-mile long. To keep vehicles off the track, the city plans to build a fence around the track. The fenced area will be square and the fence, at its closest point, will be 5 feet from the edge of the circular track. How many feet of fencing will be needed? Give your answer to the nearest foot and ignore any gates or openings. (1 mile = 5280 feet)

► Solution

Refer to the sketch at right.



First find the diameter of the track. The circumference of the track is $5280/4$, or 1320 feet. Using $C = \pi d$, solve for d to get $d = C/\pi$. So, $d = 1320/\pi$, or $d \approx 420.2$ feet. Each side of the square fence is 10 feet longer than the diameter of the track. Therefore, each side of the square must be about 430.2 feet. The amount of fencing needed is about $4 \cdot 430.2 = 1720.8$ feet, or to the nearest foot, 1721 feet.

6.7

Arc Length

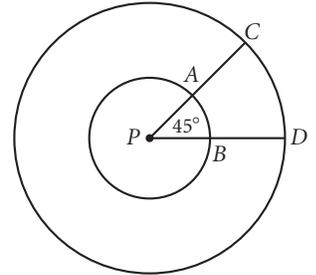
In this lesson you will

- Learn the difference between **arc length** and **arc measure**
- Find a method for calculating arc length
- Solve problems involving arc length

In the figure at right, \widehat{AB} and \widehat{CD} have the same *measure* (45°) because their central angles, $\angle APB$ and $\angle CPD$, are the same angle. However, the *lengths* of \widehat{AB} and \widehat{CD} are clearly different— \widehat{CD} is longer than \widehat{AB} . In this lesson you'll learn how to find the length of an arc.

Read Example A in your book. It illustrates that if an arc has measure x , then the arc is $\frac{x}{360^\circ}$ of the circle. For example, each of the 45° arcs in the figure at right is $\frac{45^\circ}{360^\circ}$, or $\frac{1}{8}$, of a complete circle.

Think about how the fractions in Example A are related to the length of the arc. If you travel a quarter of the way around a circle, you cover $\frac{1}{4}$ of its circumference. If you travel halfway around, you cover $\frac{1}{2}$ the circumference. The length of an arc, or **arc length**, is a fraction of the circumference of its circle. While arc measure is expressed in degrees, arc length is expressed in a unit of distance.



Investigation: Finding the Arcs

Look at the figures in Step 1 in your book. Find the fraction of the circle each arc makes up. You should get the following results.

\widehat{AB} is $\frac{1}{4}$ of circle T

\widehat{CED} is $\frac{1}{2}$ of circle O

\widehat{GH} is $\frac{140^\circ}{360^\circ}$, or $\frac{7}{18}$, of circle P

Now find the circumference of each circle using the formula $C = \pi d$ or $C = 2\pi r$. For example, because circle P has radius 36 feet, its circumference is $2\pi(36)$, or 72π feet.

Use the fractions you found in Step 1 and the circumference of each circle to find the length of each arc. For example, \widehat{GH} makes up $\frac{7}{18}$ of circle P , and circle P has a circumference of 72π . So,

$$\text{Length of } \widehat{GH} = \frac{7}{18}(72\pi) = 28\pi$$

The length of \widehat{GH} is 28π feet.

Generalize this method for finding the length of an arc as a conjecture.

Arc Length Conjecture The length of an arc equals the circumference times **C-66**
the arc measure divided by _____.

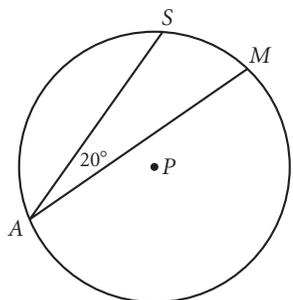
(continued)

Lesson 6.7 • Arc Length (continued)

Examples B and C in your book show how you can apply the Arc Length Conjecture. Read these examples carefully. Then read the example below.

EXAMPLE

The length of \widehat{SM} is 6π centimeters. What is the radius of circle P ?



► Solution

$m\angle SAM = 20^\circ$, so $m\widehat{SM} = 40^\circ$ by the Inscribed Angle Conjecture. This means that \widehat{SM} is $\frac{40^\circ}{360^\circ}$, or $\frac{1}{9}$, of the circumference of the circle.

Arc length $\widehat{SM} = \frac{1}{9}C$ Use the Arc Length Conjecture.

$$6\pi = \frac{1}{9}C \quad \text{Substitute } 6\pi \text{ for the arc length.}$$

$$54\pi = C \quad \text{Multiply both sides by 9.}$$

The circumference is 54π centimeters. Use the circumference formula to find the radius.

$$C = 2\pi r \quad \text{The circumference formula.}$$

$$54\pi = 2\pi r \quad \text{Substitute } 54\pi \text{ for the arc length.}$$

$$27 = r \quad \text{Divide both sides by } 2\pi.$$

The radius is 27 centimeters.