In this lesson, you

- Learn about three types of rigid transformations—translation, rotation, and reflection
- Use patty paper to model reflection
- Learn how to identify figures with rotational symmetry or reflectional symmetry

In your book, read the text before the investigation very carefully. Below is a summary of some of the key points.

1. A transformation that creates an image that is congruent to the original figure is called a rigid transformation, or isometry. Three types of rigid transformations are translation, rotation, and reflection.

2. A transformation that changes the size or shape of a figure is a nonrigid transformation.

3. A translation slides a figure along a straight-line path, moving each point the same distance in the same direction. You can describe a translation using a translation vector, which specifies both the distance and the direction.

4. A rotation turns a figure about a fixed point, rotating each point the same number of degrees. You can describe a rotation by giving the center point, the number of degrees, and the direction (clockwise or counterclockwise). When a direction is not specified, the rotation is assumed to be counterclockwise.

5. A reflection flips a figure over a line, creating the mirror image of the figure. You can describe a reflection by specifying the line of reflection.

Investigation: The Basic Property of a Reflection

Follow Steps 1 and 2 in your book to create a figure and its reflected image. Then, draw segments connecting each vertex with its image point.

Measure the angles where the segments intersect the line of reflection. What do you find? The line of reflection divides each connecting segment into two smaller segments. How do the lengths of the smaller segments compare? Use your findings to complete this conjecture.

Reflection Line Conjecture The line of reflection is the ______________ of every segment joining a point in the original figure with its image.
Lesson 7.1 • Transformations and Symmetry (continued)

Read the remaining text in this lesson. Below is a summary of some of the key points.

1. A figure that can be reflected over a line so that the resulting image coincides with the original has **reflectional symmetry**. The reflection line is called a **line of symmetry**. A figure can have more than one line of symmetry.

2. A figure that can be rotated *less than a full turn* about a point so that the rotated image coincides with the original has **rotational symmetry** or **point symmetry**. If the image coincides with the original figure $n$ times during a full turn, then the figure is said to have $n$-fold rotational symmetry.

**EXAMPLE** Describe all the symmetries of an equilateral triangle.

**Solution** An equilateral triangle has three reflectional symmetries. There is a line of reflection through each vertex and the midpoint of the opposite side.

An equilateral triangle has three rotational symmetries (that is, it has 3-fold rotational symmetry). It can be rotated $120^\circ$, $240^\circ$, and $360^\circ$ about its center, and it will coincide with itself.
Properties of Isometries

In this lesson, you

- Use ordered pair rules to transform polygons in the coordinate plane
- Learn the ordered pair rules that correspond to various isometries
- Discover how to determine the minimal path from a point to a line to another point on the same side of the line

You can use an ordered pair rule to transform figures in the coordinate plane. An ordered pair rule describes how each point in an original figure is relocated to create an image.

Example A in your book illustrates that the rule \((x, y) \rightarrow (x + 2, y - 3)\) is a translation that moves each point of a figure right 2 units and down 3 units. Read Example A carefully. In general, the rule \((x, y) \rightarrow (x + h, y + k)\) is a translation of \(h\) units horizontally and \(k\) units vertically.

Investigation 1: Transformations on a Coordinate Plane

In this investigation, you will explore four ordered pair rules.

Follow Steps 1 and 2 in your book. Draw your original polygon in Quadrant I, II, or IV. (Here are some examples in which the original polygon is in Quadrant III.)

Using your drawings and some patty paper, determine whether each transformation is a reflection, translation, or rotation. Identify the lines of reflection and centers and angles of rotation.

Now, consider the examples. In each graph, the solid figure in Quadrant III is the original, and the dashed figure is the image. (Note: The line \(y = x\) has been added to the last graph.) Determine how the original figure has been transformed to create the image. Do you get the same results you got for your polygon?
Lesson 7.2 • Properties of Isometries (continued)

Use your findings to complete the Coordinate Transformations Conjecture in your book.

Now you will revisit “poolroom geometry.” Recall that when a ball rolls without spin into a cushion, the outgoing angle is congruent to the incoming angle.

Investigation 2: Finding a Minimal Path

Follow Steps 1–4 of the investigation in your book. Then, unfold the paper and draw $AB'$. Notice that $AB'$ passes through point $C$.

Measure the length of the path from $A$ to $B'$. Measure the length of the two-part path from $A$ to $C$ to $B$. You should find that the lengths of the paths are the same.

The path from $A$ to $C$ to $B$ is the shortest path, or the minimal path, from $A$ to the cushion to $B$. To see why, choose any other point $D$ on the cushion. The path from $A$ to $D$ to $B$ is the same length as the path from $A$ to $D$ to $B'$ (why?), so it is also shorter than the path from $A$ to $D$ to $B$. Because $AB'$ is the same length as the path from $A$ to $C$ to $B$, the path from $A$ to $C$ to $B$ is shorter than the path from $A$ to $D$ to $B$. This argument is given in symbols below.

$AD + DB = AD + DB'$

$AB' < AD + DB'$

$AB' < AD + DB$

$AB' = AC + CB$

$AC + CB < AD + DB$

Complete this conjecture.

**Minimal Path Conjecture** If points $A$ and $B$ are on one side of line $\ell$, then the minimal path from point $A$ to line $\ell$ to point $B$ is found by reflecting point ________________ over line $\ell$, drawing segment ____________, then drawing segments $AC$ and ________________, where point $C$ is the point of intersection of segment ________________ and line $\ell$.

Your findings in the investigation show you that, if you want to hit a ball from point $A$ off the cushion so it passes through point $B$, you should visualize point $B$ reflected over the cushion and then aim at the reflected image.

Example B in your book applies what you learned in Investigation 2 to solve a problem about miniature golf. Read this example carefully.
In this lesson, you

- Find the single transformation equivalent to the composition of two translations
- Find the single transformation equivalent to the composition of reflections over two parallel lines
- Find the single transformation equivalent to the composition of reflections over two intersecting lines

When you apply a transformation to a figure and then apply another transformation to its image, the resulting transformation is called a composition of transformations.

In the example in your book, a figure is translated by one rule, and then its image is translated by a different rule. The composition of the two translations is equivalent to a single translation. Read this example carefully and make sure you understand it.

Investigation 1: Reflections over Two Parallel Lines

Follow Steps 1–4 to reflect a figure over one line and then reflect the image over a second line, parallel to the first.

What type of transformation would take the original figure to the final image? (Hint: How does the orientation of the final image compare with the orientation of the original?)

Use a compass or patty paper to compare the distance between the parallel lines to the distance between a point on the original figure and the corresponding point on the final image.

Use your findings to complete this conjecture.

**Reflections over Parallel Lines Conjecture** A composition of two reflections over two parallel lines is equivalent to a single ___________.
In addition, the distance from any point to its second image under the two reflections is ______________ the distance between the parallel lines.

The next example is Exercise 3 in your book.
Lesson 7.3 • Compositions of Transformations (continued)

**EXAMPLE**

Lines $m$ and $n$ are parallel and 10 cm apart.

a. Point $A$ is 6 cm from line $m$ and 16 cm from line $n$.

Point $A$ is reflected over line $m$, and then its image, $A'$, is reflected over line $n$ to create a second image, point $A''$.

How far is point $A$ from point $A''$?

b. What if $A$ is reflected over $n$, and then its image is reflected over $m$? Find the new image and distance.

**Solution**

a. By the Reflections over Parallel Lines Conjecture, the distance between $A$ and $A''$ is 20 cm, twice the distance between the lines. A drawing verifies this.

b. By the Reflections over Parallel Lines Conjecture, the distance between $A$ and $A''$ is 20 cm. A drawing verifies this.

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Investigation 2: Reflections Over Two Intersecting Lines

Follow Steps 1–4 to reflect a figure over one line and then reflect the image over a second line that intersects the first.

Draw two rays that start at the point of intersection of the two lines and that pass through corresponding points on the original figure and its second image. What single transformation would take the original figure to the final image?

You should have found that the two reflections are equivalent to a single rotation. Use a protractor to compare the angle of rotation (that is, the angle created by the two rays) with the acute angle formed by the intersecting lines.

Use your findings to complete this conjecture.

**Reflections over Intersecting Lines Conjecture**

A composition of two reflections over a pair of intersecting lines is equivalent to a single _______.

The angle of ____________ is ____________ the acute angle between the pair of intersecting reflection lines.

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There are many other ways to combine transformations. For example, the composition of a translation and a reflection is called a glide reflection.

Page 376 of your book shows examples of glide reflection.
In this lesson, you

- Learn about the three regular tessellations
- Discover all the possible semiregular tessellations

An arrangement of shapes that covers a plane completely without gaps or overlaps is called a tessellation. Read the text in your book before the investigation. Here is a summary of some of the main points.

1. For shapes to create a tessellation, their angles, when arranged around a point, must have measures that add to exactly 360°.
2. A tessellation that uses only one shape is called a monohedral tiling.
3. A monohedral tessellation of congruent regular polygons is called a regular tessellation. The only polygons that create a regular tessellation are equilateral triangles, squares, and regular hexagons. (These are the regular polygons with angle measures that are factors of 360°.)
4. When the same combination of two or more regular polygons meet in the same order at each vertex of a tessellation, it is called a semiregular tessellation.
5. You can describe a tessellation by giving its vertex arrangement, or numerical name. To name a tessellation, list the number of sides of each shape, in order as you move around a vertex. For example, each vertex of the tessellation at right is surrounded by a square (4 sides), a hexagon (6 sides), and a dodecagon (12 sides). So, the numerical name for this tessellation is 4.6.12.

Investigation: The Semiregular Tessellations

There are eight different semiregular tessellations. Your book shows three (4.8.8, 4.6.12, and 3.12.12). In this investigation, you will find the other five. All five are made from combinations of triangles, squares, and hexagons.

You will need triangles, squares, and hexagons either from a set of pattern blocks or traced or copied from the set below and cut out. If you have geometry software available, you may use that instead.

First, look for combinations of two polygons that can be used to create a semiregular tessellation. Start by finding combinations of angle measures that add to 360°. For example, because $4 \cdot 60° + 120° = 360°$, four triangles and one hexagon could be arranged around a vertex. Try to find a way to arrange the shapes so the pattern can be continued indefinitely. (Remember, the polygons must meet in the same order at each vertex.)
Lesson 7.4 • Tessellations with Regular Polygons (continued)

Here is the tessellation, labeled with its numerical name.

![Tessellation Diagram]

You should be able to find four semiregular tessellations that can be made with two different polygons. Sketch each one and label it with its numerical name.

Now, look for combinations of three polygons that can be used to create a semiregular tessellation. Again, first find combinations of angle measures that add to 360°, and then see if you can make a tessellation. You should find one semiregular tessellation that can be made with three different polygons. Sketch it and label it with its numerical name.

Read the remaining text in this lesson. Here is a summary of the key points.

1. The three regular and eight semiregular tessellations are called the Archimedean tilings.

2. The regular and semiregular tessellations are also called 1-uniform tilings because all the vertices are identical. A tessellation with two types of vertices is called 2-uniform, a tessellation with three types of vertices is called 3-uniform, and so on. (See the illustrations in your book for examples.)
In this lesson, you

- Determine whether all triangles tessellate
- Determine whether all quadrilaterals tessellate
- Look at some examples of pentagonal tessellations

In Lesson 7.4, you investigated tessellations that were formed from regular polygons. Now, you will try to create tessellations from nonregular polygons.

**Investigation 1: Do All Triangles Tessellate?**

Follow the steps for “Making Congruent Triangles” on page 384 in your book to create and label 12 congruent scalene triangles. Try to form a tessellation using the triangles. Here is an example.

Look at your tessellation and the tessellation below. You should find that, in both cases, each angle fits twice around each vertex point.

The sum of the measures of the angles of a triangle is 180°. Because each angle fits twice around each point, the sum of the measures of the angles around each point is 2(180°), or 360°. As you saw in Lesson 7.4, this is the sum of the angles surrounding each vertex in any tessellation.

Do you think you would be able to create a tessellation using an isosceles triangle? Try it and see.

Your findings from this investigation lead to the following conjecture.

**Tessellating Triangles Conjecture** Any triangle will create a monohedral tessellation.

You know that squares and rectangles can tile a plane, and you can probably visualize tiling with parallelograms. Will any quadrilateral tessellate? You will explore this question in the next investigation.

(continued)
Lesson 7.5 • Tessellations with Nonregular Polygons (continued)

Investigation 2: Do All Quadrilaterals Tessellate?

Create 12 congruent quadrilaterals (not parallelograms or trapezoids), and label the corresponding angles in each quadrilateral $a$, $b$, $c$, and $d$. Try to form a tessellation using the quadrilaterals. Here is an example.

Look at your tessellation and the tessellation at right. You should find that, in both cases, each angle fits once around each vertex point. This makes sense because the sum of the angle measures of a quadrilateral is $360^\circ$. Your findings lead to this conjecture.

**Tessellating Quadrilaterals Conjecture** Any quadrilateral will create a monohedral tessellation.

You know that a regular pentagon will not tessellate. However, it is possible to create tessellations from other types of pentagons. Pages 385 and 386 of your book show some examples. So far, 14 types of pentagons have been shown to tessellate. No one currently knows if there are more.

In the next example, you will create a pentagonal tessellation. This example is Exercise 2 in your book.

**EXAMPLE** Produce a pentagonal tessellation by creating the dual of the semiregular tessellation at right.

**Solution** Recall that the dual is created by connecting the centers of the polygons that surround each vertex point. Here is the result.