In this lesson you will

- learn about the trigonometric ratios associated with a right triangle
- use trigonometric ratios to find unknown side lengths in a right triangle
- use trigonometric inverses to find unknown angle measures in a right triangle

Suppose you fly a kite. There is a strong wind, so the string is pulled taut. You have marked the string, so you know how much string has been let out, and you can measure the angle the string makes with the horizontal. You can use a trigonometric ratio to find the kite’s height. In this lesson you will learn how.

Trigonometry relates the angle measures in right triangles to the side lengths. First, recall that triangles with the same angle measures are similar, and so the ratios of corresponding sides are equal. In right triangles, there are special names for the ratios.

For any acute angle \( A \) in a right triangle, the sine of \( \angle A \) is the ratio of the length of the leg opposite \( \angle A \) to the length of the hypotenuse.

\[
\sin A = \frac{\text{opposite leg}}{\text{hypotenuse}} = \frac{a}{c}
\]

The cosine of \( \angle A \) is the ratio of the length of the leg adjacent to \( \angle A \) to the length of the hypotenuse.

\[
\cos A = \frac{\text{adjacent leg}}{\text{hypotenuse}} = \frac{b}{c}
\]

The tangent of \( \angle A \) is the ratio of the length of the opposite leg to the length of the adjacent leg.

\[
\tan A = \frac{\text{opposite leg}}{\text{adjacent leg}} = \frac{a}{b}
\]

Read Example A in your book, and then read the example below.

**EXAMPLE**

Find the unknown length, \( c \).

**Solution**

You know the length of the side opposite to the 25° angle, and you want to find the length of the hypotenuse. Therefore, you can use the sine ratio.

\[
\sin 25° = \frac{14}{c}
\]

\[
c = \frac{14}{\sin 25°} \approx 33.13
\]
Lesson 12.1 • Right Triangle Trigonometry (continued)

The inverse of a trigonometric function gives the measure of the angle that has a given ratio. For example, \( \sin 30^\circ = \frac{1}{2} \), so \( \sin^{-1}\left(\frac{1}{2}\right) = 30^\circ \). Example B in your book uses the inverse tangent function. Read the example carefully.

Investigation: Steep Steps

Read the opening paragraph of the investigation in your book. Complete Steps 1–4 of the investigation and then compare your answers to those below.

**Step 1**  First sketch a step with the maximum rise and minimum run.

Let the angle of inclination be \( x \). Because the floor and the run are both horizontal (and thus parallel), the angle between the run and the hypotenuse is also \( x \). You know the lengths of the opposite and adjacent sides, so use tangent to solve for \( x \).

\[
\tan x = \frac{7.75}{10}
\]

\[
x = \tan^{-1}\left(\frac{7.75}{10}\right) = 37.8^\circ
\]

The angle of inclination is about 38°.

**Step 2**  Two sets of stairs that will fit both the code and the rule of thumb are a set with a unit run of 11 in. and a unit rise of 6.5 in. and a set with a unit run of 11.5 in. and a unit rise of 6 in. The respective angles of inclination for these sets are given by \( \tan^{-1}\left(\frac{6.5}{11}\right) = 30.6^\circ \) and \( \tan^{-1}\left(\frac{6}{11.5}\right) = 27.6^\circ \).

An example of a set of stairs that fits the rule of thumb but does not fit the code is one with a unit rise of about 8.75 in. and a unit run of 8.75 in. The angle of inclination for this set is given by \( \tan^{-1}\left(\frac{8.75}{8.75}\right) = 45.0^\circ \).

**Step 3**  Refer to the photo and diagram on page 682 of your book.

a. There are infinitely many designs possible, but not all designs will meet the code given in Step 1. For example, a stair with a unit rise of about 15.6 in. and a unit run of about 41 in. would fit the 20.8° angle of inclination but would not fit the code, because the rise is too high.

b. To find the solution, let the unit run be represented by \( r \). Then the unit rise will be represented by \( 17.5 - r \). To find \( r \), use the tangent ratio.

\[
\tan 20.8^\circ = \frac{17.5 - r}{r}
\]

\[
0.3799 = \frac{17.5 - r}{r}
\]

\[
0.3799r = 17.5 - r
\]

\[
1.3799r = 17.5
\]

\[
r = \frac{17.5}{1.3799}
\]

\[
r = 12.68 \text{ in.}
\]

So the run is 12.68 in. and the rise is \( 17.5 - 12.68 = 4.82 \) in.

**Step 4**  Use the tangent function and let \( x \) be the angle of inclination. Using \( \tan x = \frac{1}{16} \), \( x = \tan^{-1}\left(\frac{1}{16}\right) = 3.58^\circ \) and using \( \tan x = \frac{1}{20} \), \( x = \tan^{-1}\left(\frac{1}{20}\right) = 2.86^\circ \).

So the angle should be between 2.86° and 3.58°.
In this lesson you will

- discover and apply the Law of Sines, which describes a relationship between the sides and angles of an oblique triangle

You have investigated the relationships between the sides and angles of right triangles. Now you will investigate relationships between the sides and angles of nonright, or oblique, triangles.

**Investigation: Oblique Triangles**

**Step 1** Draw an acute triangle $ABC$. Label the side opposite $\angle A$ as $a$, the side opposite $\angle B$ as $b$, and the side opposite $\angle C$ as $c$. Then, draw the altitude from $\angle A$ to $BC$. Label the height $h$. At right is one example.

**Step 2** From this diagram, you can write the following equations:

$$\sin B = \frac{h}{c}, \text{ or } h = c \sin B$$

$$\sin C = \frac{h}{b}, \text{ or } h = b \sin C$$

Because both $c \sin B$ and $b \sin C$ are equal to $h$, they are equal to each other. That is,

$$c \sin B = b \sin C$$

Dividing both sides of the equation above by $bc$ gives

$$\frac{\sin B}{b} = \frac{\sin C}{c}$$

**Step 3** Now, draw the altitude from $\angle B$ to $AC$ and label the height $j$. Using a method similar to that in Step 2, you should find that

$$\frac{\sin A}{a} = \frac{\sin C}{c}$$

(Make sure you can derive this equation on your own!)

**Steps 4 and 5** You can combine the proportions from Steps 2 and 3 to write an extended proportion:

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

The triangle you drew in Step 1 was acute. Do you think the same proportion will be true for obtuse triangles?

**Step 6** Draw an obtuse triangle $ABC$ and measure each angle and side. At right is one example.

Find $\frac{\sin A}{a}$, $\frac{\sin B}{b}$, and $\frac{\sin C}{c}$ for your triangle. For the triangle at right:

$$\frac{\sin A}{a} = \frac{\sin 31^\circ}{4} = 0.13 \quad \frac{\sin B}{b} = \frac{\sin 23^\circ}{3} = 0.13 \quad \frac{\sin C}{c} = \frac{\sin 126^\circ}{6.3} = 0.13$$

So, it appears that $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$ holds for obtuse triangles as well.

(continued)
Lesson 12.2 • The Law of Sines (continued)

Example A in your book applies what you learned in the investigation to a real-world problem. Read the example carefully. The relationship you discovered in the investigation is called the Law of Sines. It is summarized in the “Law of Sines” box in your book.

Example B shows how to apply the Law of Sines to find an unknown side length in a triangle when you know the measures of two angles and the length of one side. Read the example carefully. Test your understanding by finding the length of side $\overline{AC}$. (Hint: You’ll need to find the measure of $\angle B$ first.) You should find that the length of $\overline{AC}$ is about 15.4 cm.

You can also use the Law of Sines to find an unknown angle measure when you know two side lengths and the measure of the angle opposite one of the sides. However, in this case you may find more than one solution. To help you understand why there may be more than one solution, look at the diagrams on page 693 of your book and read Example C. Here is another example.

**EXAMPLE**

In $\triangle ABC$, the measure of $\angle A$ is 30°, the length of side $\overline{AB}$ is 8 cm, and the length of side $\overline{BC}$ is 5 cm. Sketch and label two triangles that fit this description. For each triangle, find the measures of $\angle B$ and $\angle C$ and the length of side $\overline{AC}$.

**Solution**

The two possibilities are shown below.

To find one possible measure for $\angle C$, use the Law of Sines.

\[
\frac{\sin 30^\circ}{5} = \frac{\sin C}{8}
\]

\[
\sin C = \frac{8 \sin 30^\circ}{5}
\]

\[
C = \sin^{-1} \left( \frac{8 \sin 30^\circ}{5} \right) = 53.1^\circ
\]

The measure of $\angle C$ is 53.1°, so the measure of $\angle B$ is $180^\circ - (30^\circ + 53.1^\circ)$, or 96.9°. To find the length of $\overline{AC}$, use the Law of Sines again.

\[
\frac{\sin 30^\circ}{5} = \frac{\sin 96.9^\circ}{b}
\]

\[
b = \frac{5 \sin 96.9^\circ}{\sin 30^\circ} = 9.9 \text{ cm}
\]

The length of $\overline{AC}$ is 9.9 cm.

The other possible measure for $\angle C$ is the supplement of 53.1°, or 126.9°. The measure of $\angle B$ is then $180^\circ - (30^\circ + 126.9^\circ)$, or 23.1°. Use the Law of Sines to find the length of $\overline{AC}$.

\[
\frac{\sin 30^\circ}{5} = \frac{\sin 23.1^\circ}{b}
\]

\[
b = \frac{5 \sin 23.1^\circ}{\sin 30^\circ} = 3.9 \text{ cm}
\]

The length of $\overline{AC}$ is 3.9 cm.
In this lesson you will

- use the **Law of Cosines** to find unknown measures of a triangle when you know two side lengths and the measure of the included angle
- use the **Law of Cosines** to find unknown measures of a triangle when you know three side lengths

You can use the Law of Sines to find side lengths or angle measures of a triangle if you know either two angle measures and one side length or two side lengths and the measure of the angle opposite one of those sides.

In Example A in your book, you are given two side lengths and the measure of the angle between the sides, and you must find the length of the third side. The Law of Sines cannot be applied in this situation. Work through the solution to see how to find the unknown side length.

If you use the procedure in Example A in a general case where you are given two side lengths, $a$ and $b$, of a triangle, $ABC$, and the measure of the included angle, $C$, you get the **Law of Cosines**:

$$c^2 = a^2 + b^2 - 2ab \cos C$$

where $c$ is opposite $\angle C$. Notice that this looks like the Pythagorean Theorem with an extra term, $-2ab \cos C$. (In fact, if $C$ is a right angle, then $\cos C$ is 0 and the equation becomes the Pythagorean Theorem.). Read the text in the “Law of Cosines” box on page 699 in your book and study the diagrams after the box.

**Investigation: Around the Corner**

Read the investigation in your book. If you have the materials and some people to help you, complete the investigation. If not, you can use the diagram at right. Complete the investigation on your own, and then compare your results to those given.

You know the lengths of two sides and the measure of an included angle, so you can use the Law of Cosines to find the length of the third side.

$\begin{align*}
c^2 &= a^2 + b^2 - 2ab \cos C & \text{The Law of Cosines.} \\
c^2 &= 2.5^2 + 2^2 - 2(2.5)(2) \cos 43^\circ & \text{Substitute the known values.} \\
c^2 &= 6.25 + 4 - 10 \cos 43^\circ & \text{Multiply.} \\
c &= \sqrt{10.25 - 10 \cos 43^\circ} & \text{Solve for } c. \\
c &\approx 1.71 & \text{Evaluate.}
\end{align*}$

The two “towns” are about 1.71 meters apart.
Lesson 12.3 • The Law of Cosines (continued)

To find the unknown measures in Example B, the Law of Cosines is applied twice. Try to find the unknown measures yourself, and then read the solution. In both the investigation and Example B, you are given two side lengths and the measure of the included angle. You can also use the Law of Cosines if you know three side lengths. The example below shows you how.

**EXAMPLE**

Find the angle measures.

\[
\begin{align*}
\text{A} & \quad 5.1 \text{ cm} \\
\text{B} & \quad 3.5 \text{ cm} \\
\text{C} & \quad 2.0 \text{ cm}
\end{align*}
\]

**Solution**

Start by using the Law of Cosines by finding the measure of \( \angle C \).

\[
c^2 = a^2 + b^2 - 2ab \cos C
\]

The Law of Cosines.

\[
3.5^2 = 5.1^2 + 2.0^2 - 2(5.1)(2.0) \cos C
\]

Substitute the known values.

\[
12.25 = 30.01 - 20.4 \cos C
\]

Multiply.

\[
-17.76 = -20.4 \cos C
\]

Subtract 30.01 from both sides.

\[
\cos C = \frac{-17.76}{20.4}
\]

Solve for \( \cos C \).

\[
C = \cos^{-1}\left(\frac{-17.76}{20.4}\right)
\]

Take the inverse cosine of both sides.

\[
C = 29.5^\circ
\]

Evaluate.

Now, use the Law of Sines to find the measure of \( \angle B \).

\[
\frac{\sin C}{c} = \frac{\sin B}{b}
\]

The Law of Sines.

\[
\frac{\sin 29.5^\circ}{3.5} = \frac{\sin B}{2.0}
\]

Substitute the known values.

\[
\sin B = \frac{2.0 \sin 29.5^\circ}{3.5}
\]

Solve for \( \sin B \).

\[
B = \sin^{-1}\left(\frac{2.0 \sin 29.5^\circ}{3.5}\right)
\]

Take the inverse sine of both sides.

\[
B \approx 16.3^\circ
\]

Evaluate.

To find the measure of \( \angle A \), use the fact that the sum of the angle measures of a triangle is 180°.

\[
A = 180^\circ -(29.5^\circ + 16.3^\circ) = 134.2^\circ
\]

Read the remainder of the lesson in your book, which summarizes what you have learned in this and the previous lesson.
In this lesson you will
- extend the definitions of sine, cosine, and tangent to include angles of any measure
- find the sine, cosine, and tangent of angles of rotation
- use reference angles to find the sine, cosine, and tangent of related angles

In Lesson 12.1, the definitions given for sine, cosine, and tangent applied to acute angles in right triangles. In this lesson, you will extend the definitions to apply to any size angle. Remember that angles in the coordinate plane are measured starting from the positive x-axis and moving counterclockwise through Quadrants I, II, III, and IV.

**Investigation: Extending Trigonometric Functions**

Read the Procedure Note and study the example shown for Step 1. Then work through the investigation in your book. After you're finished, compare your answers to the results below. Make sure your calculator is set to degrees.

**Step 1** The sample answers use the point (4, 0) as the starting point for each angle. Your answers for the coordinates and the length of the segment will vary depending on the starting point you chose, but your results for the sine, cosine, and tangent should match these results.

**a.**

\[ \sin 135^\circ = 0.707, \cos 135^\circ = -0.707, \text{ and } \tan 135^\circ = -1. \]

The coordinates of the rotated point are about \((-2.8, 2.8)\). The length of the segment is about \(\sqrt{(-2.8)^2 + 2.8^2} \approx 3.96\) units.

**b.**

\[ \sin 210^\circ = -0.5, \cos 210^\circ = -0.866, \text{ and } \tan 210^\circ = 0.577. \]

The coordinates of the rotated point are about \((-3.5, -2)\). The length of the segment is about \(\sqrt{(-3.5)^2 + (-2)^2} \approx 4.03\) units.

(continued)
Lesson 12.4 • Extending Trigonometry (continued)

c. 

\[ \sin 270^\circ = -1, \cos 270^\circ = 0, \text{ and } \tan 270^\circ \text{ is undefined. The coordinates of the } \]

rotated point are \((0, -4)\). The length of the segment is \(\sqrt{0^2 + (-4)^2} = 4\) units.

d. 

\[ \sin 320^\circ = -0.643, \cos 320^\circ = 0.766, \text{ and } \tan 320^\circ = -0.839. \text{ The coordinates of the rotated point are about } (3.1, -2.6) \]

The length of the segment is about \(\sqrt{3.1^2 + (-2.6)^2} \approx 4.05\) units.

e. 

\[ \sin -100^\circ = -0.985, \cos -100^\circ = -0.174, \text{ and } \tan -100^\circ = 5.671. \text{ The coordinates of the rotated point are about } (-0.7, -3.9) \]

The length of the segment is about \(\sqrt{(-0.7)^2 + (-3.9)^2} \approx 3.96\) units.

**Step 2** The results are summarized below. From these results you might hypothesize that sine is \(\frac{y\text{-coordinate}}{\text{segment length}}\), cosine is \(\frac{x\text{-coordinate}}{\text{segment length}}\), and tangent is \(\frac{y\text{-coordinate}}{x\text{-coordinate}}\).

<table>
<thead>
<tr>
<th>Angle</th>
<th>Sine</th>
<th>Cosine</th>
<th>Tangent</th>
</tr>
</thead>
<tbody>
<tr>
<td>135°</td>
<td>[\frac{2.8}{3.96} \approx 0.71]</td>
<td>[\frac{-2.8}{3.96} \approx -0.71]</td>
<td>[\frac{2.8}{-2.8} = 1]</td>
</tr>
<tr>
<td>210°</td>
<td>[\frac{-2}{4.03} \approx -0.50]</td>
<td>[\frac{-3.5}{4.03} \approx -0.87]</td>
<td>[\frac{-2}{-3.5} = 0.57]</td>
</tr>
<tr>
<td>270°</td>
<td>[\frac{-4}{4} = -1]</td>
<td>[\frac{0}{4} = 0]</td>
<td>is undefined</td>
</tr>
<tr>
<td>320°</td>
<td>[\frac{-2.6}{4.05} \approx -0.64]</td>
<td>[\frac{3.1}{4.05} \approx 0.77]</td>
<td>[\frac{-2.6}{3.1} = -0.84]</td>
</tr>
<tr>
<td>-100°</td>
<td>[\frac{-3.9}{3.96} \approx -0.98]</td>
<td>[\frac{-0.7}{3.96} \approx -0.18]</td>
<td>[\frac{-3.9}{-0.7} = 5.6]</td>
</tr>
</tbody>
</table>
Lesson 12.4 • Extending Trigonometry (continued)

Step 3

The length of the segment is $\sqrt{(-3)^2 + 1^2} = \sqrt{10}$. Using the method from Step 2, $\sin A = \frac{1}{\sqrt{10}}$, $\cos A = \frac{-3}{\sqrt{10}}$, and $\tan A = \frac{1}{-3}$. The calculator gives $\sin^{-1} \left( \frac{1}{\sqrt{10}} \right) \approx 18.43^\circ$ and $\tan^{-1} \left( \frac{1}{-3} \right) \approx 18.43^\circ$. This angle is in Quadrant I, so it doesn’t match the diagram. However, using the calculator, $\cos^{-1} \left( \frac{-3}{\sqrt{10}} \right) \approx 161.57^\circ$. This angle appears to match the diagram.

Step 4  The definitions are located in the definition box on page 707 of your book. Read these definitions carefully.

Read the paragraph before Example A, and then work through Examples A and B in your book. If you need to review special right triangles, read Refreshing Your Skills for Chapter 12 in your book. Below is another example similar to Example A.

**EXAMPLE**  Find the sine, cosine, and tangent of $150^\circ$ without a calculator.

**Solution**  Rotate a point counterclockwise $150^\circ$ from the positive $x$-axis. The image of the point is in Quadrant II, $30^\circ$ above the $x$-axis. The reference angle is $30^\circ$. The sine, cosine, and tangent of a $30^\circ$ reference angle are, respectively, $\frac{1}{2}$, $\frac{\sqrt{3}}{2}$, and $\frac{1}{\sqrt{3}}$.

Because the $x$-coordinate is negative and the $y$-coordinate is positive in Quadrant II, $\sin 150^\circ = \frac{1}{2}$, $\cos 150^\circ = -\frac{\sqrt{3}}{2}$, and $\tan 150^\circ = -\frac{1}{\sqrt{3}}$. 
In this lesson you will

- understand **vectors** as directed distances
- represent addition, subtraction, and **scalar** multiplication of vectors
- use vectors to solve problems
- convert vectors from one form to another

Some quantities, such as distance, velocity, and acceleration, can have directions associated with them. These directed quantities can be represented by **vectors**, which can be thought of as directed line segments. The line segment has a length, called the **magnitude**, and a **direction**. You can represent vectors as a segment with an arrowhead at one end, called the **head** or **tip**. The **tail** is the other end of the vector. Vectors can be represented in several ways. The **polar form** of a vector gives the magnitude and the angle the vector makes with the positive x-axis. For example, \( \langle 3 \angle 150^\circ \rangle \) represents a vector 3 units long directed 150° counterclockwise from the positive x-axis. The **rectangular form** of a vector gives the horizontal and vertical change from the tail to the head. For example, \( \langle -\frac{3\sqrt{3}}{2}, \frac{3}{2} \rangle \) represents a horizontal change of \(-\frac{3\sqrt{3}}{2}\) and a vertical change of \(\frac{3}{2}\).

Equivalent vectors have the same magnitude and direction, no matter where they are located in the coordinate plane. \( \langle 3 \angle 150^\circ \rangle \) and \( \langle -\frac{3\sqrt{3}}{2}, \frac{3}{2} \rangle \) are equivalent vectors.

The investigation explores some of the properties of vector addition and subtraction. Note that \( \vec{a} \) and \( a \) are two ways to designate a vector. In the equation \( a + b = c \), the boldface letters \( a, b, \) and \( c \) represent vectors, and \( c \) is the **resultant vector** of the calculation.

**Investigation: Vector Addition and Subtraction**

Work through the whole investigation in your book, and then compare your results to those below.

**Steps 1–3**

![Vector Diagram]

The rectangular form of \( c \) is \( (6, 4) \).

**Step 4**

i. 

![Vector Diagram]

The rectangular form of \( c \) is \( (6, 4) \).

ii. 

![Vector Diagram]

The rectangular form of \( c \) is \( (0, 1) \).

(continued)
Lesson 12.5 • Introduction to Vectors (continued)

The rectangular form of \( c \) is \( (3, -1) \). The rectangular form of \( c \) is \( (5, 2) \).

**Step 5** If \( a = \langle a_1, a_2 \rangle \) and \( b = \langle b_1, b_2 \rangle \), then the sum \( a + b \) is \( \langle a_1, a_2 \rangle + \langle b_1, b_2 \rangle = \langle a_1 + b_1, a_2 + b_2 \rangle \).

**Step 6**

\[ \begin{align*}
\text{i.} & \\
\text{ii.} & \\
\text{iii.} & \\
\text{iv.} & \\
\end{align*} \]

**Step 7** If \( a = \langle a_1, a_2 \rangle \) and \( b = \langle b_1, b_2 \rangle \), then the difference \( a - b \) is \( \langle a_1, a_2 \rangle - \langle b_1, b_2 \rangle = \langle a_1 - b_1, a_2 - b_2 \rangle \).

**Step 8** If \( a = \langle a_1, a_2 \rangle \) and \( k \) is a scalar, then the product \( k \cdot a \) is \( k \cdot \langle a_1, a_2 \rangle = \langle k \cdot a_1, k \cdot a_2 \rangle \).

**Step 9** The magnitudes of \( a \) and \( b \) are \( |a| = \sqrt{a_1^2 + a_2^2} = \sqrt{13} \) and \( |b| = \sqrt{b_1^2 + b_2^2} = \sqrt{17} \). If \( a = \langle a_1, a_2 \rangle \), then the magnitude of \( a \), denoted \( |a| \), is \( \sqrt{a_1^2 + a_2^2} \).

Vectors are useful for representing motion. Read Example A to explore an application of vector addition.

Sometimes the polar form of a vector is more appropriate. Example B explains how to convert from rectangular form to polar form. Read Example B and make sure you understand how to convert from rectangular to polar form.

Read the text following Example B. Be sure you understand how to change a bearing to an angle that gives direction of a vector in polar form. In Example C, the vectors must be converted from polar form to rectangular form to add them. Work carefully through Example C.
In this lesson you will

- use a parameter to write parametric equations that separately define $x$ and $y$
- graph parametric equations
- use parametric equations to model real-world problems

So far, you have used equations to relate $x$ and $y$ to each other. Sometimes you want to express $x$ and $y$ as separate functions of a third variable, $t$, called the parameter. These parametric equations provide you with more information and better control over what points you plot. You can use parametric equations to express $x$- and $y$-coordinates as functions of time.

Example A in your book shows how to use parametric equations to model a motion problem. Read Example A and its solution carefully. Then read the following example.

**EXAMPLE A**

James is rowing a boat 30 ft across a river. He rows at a rate of 1 ft/s directly toward the opposite shore. The current moves perpendicular to his direction of rowing at a rate of 3 ft/s. The post where James wants to tie up his rowboat is 100 ft downstream from his starting point. Will James make it to the other side of the river before he passes the post?

**Solution**

Let $x$ represent the distance in feet the boat moves due to the current, let $y$ represent the distance in feet James has rowed across the river, and let $t$ represent the time in seconds. Then $x = 3t$ and $y = t$. Graph this pair of equations on your calculator. See Calculator Note 12C to learn how to enter and graph parametric equations. Use an appropriate window for the context.

You can picture the post at the point $(100, 30)$. If you trace a point on the graph, you will see that James will have 10 feet to spare before he reaches the post.

Parametric equations can help you model complicated situations involving motion. Many pairs of parametric equations can be written as a single equation using only $x$ and $y$. If you rewrite a parametric model as a single equation, then you’ll have two different ways to study a situation.
Lesson 12.6 • Parametric Equations (continued)

Investigation: Parametric Walk

Steps 1 and 2  Read Steps 1 and 2 and the Procedure Note of the investigation in your book. Make sure you can visualize what is going on: A segment is marked on a coordinate grid. As a person walks along the segment, one motion sensor (held by recorder X) is recording how the x-coordinate of the person’s path changes and one sensor (held by recorder Y) is recording how the y-coordinate of the person’s path changes.

Enter the sample data in your calculator and complete the rest of the investigation on your own. Then compare your results to those below.

Step 3  Use your calculator to find the median-median lines. The median-median line for the \((t, x)\) data is \(\hat{x} = -0.18t + 1.8\).

<table>
<thead>
<tr>
<th>Data collected by recorder X</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>(t)</td>
<td>(x)</td>
</tr>
<tr>
<td>0.1</td>
<td>1.78</td>
</tr>
<tr>
<td>0.6</td>
<td>1.71</td>
</tr>
<tr>
<td>1.1</td>
<td>1.62</td>
</tr>
<tr>
<td>1.6</td>
<td>1.50</td>
</tr>
<tr>
<td>2.1</td>
<td>1.43</td>
</tr>
<tr>
<td>2.6</td>
<td>1.36</td>
</tr>
<tr>
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<td>1.26</td>
</tr>
<tr>
<td>3.6</td>
<td>1.19</td>
</tr>
<tr>
<td>4.1</td>
<td>1.10</td>
</tr>
<tr>
<td>4.6</td>
<td>1.00</td>
</tr>
<tr>
<td>5.0</td>
<td>0.95</td>
</tr>
</tbody>
</table>

Step 4  The median-median line for the \((t, y)\) data is \(\hat{y} = 0.10t + 1.98\).

<table>
<thead>
<tr>
<th>Data collected by recorder Y</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>(t)</td>
<td>(y)</td>
</tr>
<tr>
<td>0.1</td>
<td>1.95</td>
</tr>
<tr>
<td>0.6</td>
<td>2.02</td>
</tr>
<tr>
<td>1.1</td>
<td>2.11</td>
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<td>2.25</td>
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<td>3.1</td>
<td>2.32</td>
</tr>
<tr>
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<td>2.38</td>
</tr>
<tr>
<td>4.1</td>
<td>2.39</td>
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<tr>
<td>4.6</td>
<td>2.47</td>
</tr>
<tr>
<td>5.0</td>
<td>2.50</td>
</tr>
</tbody>
</table>

Step 5  The graph at right shows a plot of the \((x, y)\) values, along with graphs of the parametric functions \(x = -0.18t + 1.8\) and \(y = 0.10t + 1.98\). The parametric functions seem to fit the data.

Step 6  Solving \(\hat{x} = -0.18t + 1.8\) for \(t\) gives \(t = \frac{x + 1.8}{-0.18}\). Substitute this expression for \(t\) into the equation for \(y\): \(\hat{y} = 0.10\left(x + 1.8\right) + 1.98\).

Step 7  The graph at right shows the \((x, y)\) data and the function \(\hat{y} = 0.10\left(x + 1.8\right) + 1.98\) from Step 6.

Step 8  Eliminating the parameter gives the same graph, but you lose the information about the time, and you cannot limit the values of \(t\) to show only the segment on the line that was actually walked.
Lesson 12.6 • Parametric Equations (continued)

Read the text following the investigation and Example B. Example B explains how to model projectile motion parametrically. The example that follows also concerns projectile motion.

**EXAMPLE B**  
Peter punts a football at an angle of 55° so that it has an initial velocity of 75 ft/s. If his foot contacts the ball at a height 3.5 ft above the ground, how far does the ball travel horizontally before it hits the ground?

**Solution**  
Draw a picture and find the $x$- and $y$-components of the initial velocity.

\[
\cos 55° = \frac{x}{75} \quad \sin 55° = \frac{y}{75}
\]

\[
x = 75\cos 55° \quad y = 75\sin 55°
\]

The horizontal motion is affected only by the initial speed and angle, so the horizontal distance is modeled by $x = 75t\cos 55°$.

The vertical motion is affected by the force of gravity and the initial height. Its equation is $y = -16t^2 + 75t\sin 55° + 3.5$.

To find when the ball hits the ground, find $t$ when $y$ is 0.

\[
-16t^2 + 75t\sin 55° + 3.5 = 0
\]

\[
t = \frac{-75\sin 55° \pm \sqrt{(75\sin 55°)^2 - 4(-16)(3.5)}}{2(-16)}
\]

\[
t \approx -0.056 \text{ or } t \approx 3.896
\]

Only the positive answer makes sense in this situation. The ball hits the ground about 3.896 seconds after it is kicked. To find how far the ball has traveled, substitute this $t$-value into the equation for $x$: $x = 75(3.896)\cos 55° \approx 167.6$. The ball travels about 167.6 ft, or 56 yd, horizontally.