Exponential Functions

In this lesson you will

- write a recursive formula to model radioactive decay
- find an exponential function that passes through the points of a geometric sequence
- learn about half-life for exponential decay and doubling time for exponential growth

In Chapter 1, you used recursive formulas to model geometric growth and decay. Recursive formulas generate only discrete values, such as the amount of money in an account after 1 year or 2 years. In many real-life situations, growth and decay happen continuously. In this lesson you will find explicit formulas that allow you to model continuous growth and decay.

Investigation: Radioactive Decay

Read the first paragraph and the Procedure Note in the investigation. If you have a die, you can conduct the experiment yourself by following these steps:

1. Draw 30 dots on a sheet of paper.
2. Roll the die once for each dot. If you roll a 1, erase or cross out the dot.
3. Count and record the number of dots remaining.
4. Repeat Steps 2 and 3 until fewer than three dots remain.

After you have collected the data, complete Steps 1–6 in your book. The results given below use these sample data:

<table>
<thead>
<tr>
<th>Stage</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
</tr>
</thead>
<tbody>
<tr>
<td>People standing (or dots left)</td>
<td>30</td>
<td>26</td>
<td>19</td>
<td>17</td>
<td>14</td>
<td>12</td>
<td>11</td>
<td>9</td>
<td>8</td>
<td>8</td>
<td>5</td>
<td>5</td>
<td>2</td>
</tr>
</tbody>
</table>

Step 1  See the table above.
Step 2  At right is a graph of the data. The graph resembles a decreasing geometric sequence.
Step 3  The initial term is \( u_0 = 30 \). To find the common ratio, look at the ratios of consecutive \( y \)-values:

\[
\frac{26}{30} \approx 0.867, \quad \frac{19}{26} \approx 0.731, \quad \frac{17}{19} \approx 0.895, \quad \frac{14}{17} \approx 0.824, \quad \frac{12}{14} \approx 0.857, \quad \frac{11}{12} \approx 0.917,
\]

\[
\frac{9}{11} \approx 0.818, \quad \frac{8}{9} \approx 0.889, \quad \frac{8}{8} \approx 1, \quad \frac{5}{8} \approx 0.625, \quad \frac{5}{5} \approx 1, \quad \frac{2}{5} \approx 0.4
\]

Find the mean of these 12 ratios to obtain \( r = 0.8185 \).

By completing the table in your book, you should have noticed a pattern: \( u_1 = u_0 \cdot r \), \( u_2 = u_0 \cdot r^2 \), and \( u_3 = u_0 \cdot r^3 \), and you can extend this pattern to \( u_n = u_0 \cdot r^n \). Thus the explicit formula for the data is \( u_n = 30 \cdot 0.8185^n \). (continued)
Lesson 5.1 • Exponential Functions (continued)

Step 4 The graph of the data with equation \( f(x) = 30 \cdot 0.8185^x \) is shown at right.

An equation with the same common ratio that passes through the point \((1, 26)\) is \( f(x) = 26 \cdot 0.8185^{x-1} \).

Step 5 You should experiment with different equations to find the one that you think best fits the data. Would you prefer a graph that hits more of the data points, or one with the same number of points above and below it? If possible, compare and discuss your model with other classmates.

Step 6 The equation \( f(x) = 6 \cdot r^{x-6} \) represents an exponential function with ratio \( r \) that contains the point \((6, u_6)\).

A formula for a geometric sequence generates a set of discrete points. Now, you will learn how to find the equation of a curve that passes through the points. Read the Science Connection about half-life in your book. Then work through Example A in your book and read the example below.

**EXAMPLE** A rare coin in Jo’s coin collection is now worth $450. The value has been increasing by 15% each year. If the value continues to increase at this rate, how much will the coin be worth in \(11 \frac{1}{2}\) years?

**Solution** The value is multiplied by \((1 + 0.15)\) each year:

\[
\begin{align*}
450 \cdot (1 + 0.15) & \quad \text{Value after 1 year.} \\
450 \cdot (1 + 0.15) \cdot (1 + 0.15) &= 450(1 + 0.15)^2 \quad \text{Value after 2 years.} \\
450 \cdot (1 + 0.15) \cdot (1 + 0.15) \cdot (1 + 0.15) &= 450(1 + 0.15)^3 \quad \text{Value after 3 years.} \\
450 \cdot (1 + 0.15)^n & \quad \text{Value after } n \text{ years.}
\end{align*}
\]

So, the explicit formula is \( u_n = 450(1 + 0.15)^n \). The equation of the continuous function through the points is \( y = 450(1 + 0.15)^x \).

You can use the continuous function to find the value of the coin at any time.

To find the value after \(11 \frac{1}{2}\) years, substitute 11.5 for \(x\).

\[
y = 450(1 + 0.15)^{11.5} = 2,245.11
\]

The continuous function found in the example is an exponential function, a continuous function with a variable in the exponent. Read the “Exponential Function” box and the text that follows on page 254 of your book. Then work carefully through Example B, which shows how two different transformations of an exponential function can result in the same graph.
In this lesson you will

- review the properties of exponents
- solve exponential equations and power equations

Recall that in an exponential expression $a^n$, $a$ is called the base and $n$ is called the exponent. You can say that $a$ is raised to the power of $n$. If the exponent is a positive integer, you can write the expression in expanded form. For example, $5^4 = 5 \cdot 5 \cdot 5 \cdot 5$.

In beginning algebra, you learned properties for rewriting expressions involving exponents. In this lesson, you review these properties and see how they can help you solve equations.

**Investigation: Properties of Exponents**

Complete the investigation on your own. When you are done, compare your results with those below.

**Step 1**

a. $2^3 \cdot 2^4 = (2 \cdot 2 \cdot 2) \cdot (2 \cdot 2 \cdot 2 \cdot 2) = 2^7$

b. $x^5 \cdot x^{12} = (x \cdot x \cdot x \cdot x \cdot x) \cdot (x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x) = x^{17}$

c. $10^2 \cdot 10^3 = (10 \cdot 10) \cdot (10 \cdot 10 \cdot 10 \cdot 10) = 10^7$

**Step 2** $a^m \cdot a^n = a^{m+n}$

**Step 3**

a. $\frac{4^5}{4^2} = \frac{4 \cdot 4 \cdot 4 \cdot 4 \cdot 4}{4 \cdot 4} = 4^3$

b. $\frac{x^8}{x^6} = \frac{x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x}{x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x} = x^2$

c. $\frac{(0.94)^{15}}{(0.94)^5} = \frac{(0.94) \cdot (0.94) \cdot (0.94) \cdot (0.94) \cdot (0.94) \cdot (0.94) \cdot (0.94) \cdot (0.94) \cdot (0.94) \cdot (0.94)}{(0.94) \cdot (0.94) \cdot (0.94) \cdot (0.94) \cdot (0.94) \cdot (0.94)}$

$= (0.94)^{10}$

**Step 4** $\frac{a^m}{a^n} = a^{m-n}$

**Step 5**

a. $\frac{2^3}{2^1} = \frac{1 \cdot 2 \cdot 2 \cdot 2}{2 \cdot 2} = 1^1$

b. $\frac{4^5}{4^7} = \frac{4 \cdot 4 \cdot 4 \cdot 4 \cdot 4}{4 \cdot 4 \cdot 4 \cdot 4 \cdot 4 \cdot 4} = 1^2$

c. $\frac{x^3}{x^8} = \frac{x \cdot x \cdot x}{x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x} = 1^3$

**Step 6**

a. $2^{3-4} = 2^{-1}$

b. $4^{5-7} = 4^{-2}$

c. $x^{3-8} = x^{-5}$

**Step 7** $\frac{1}{a^n} = a^{-n}$

**Step 8** One example is $(2^3)^4 = (2^3)(2^3)(2^3)(2^3) = 2^{3+3+3+3} = 2^{3\cdot 4}$. You can generalize this result as $(a^n)^m = a^{mn}$.

(continued)
Lesson 5.2 • Properties of Exponents and Power Functions (continued)

**Step 9**  One example is \((2 \cdot 3)^3 = (2 \cdot 3)(2 \cdot 3)(2 \cdot 3) = 2 \cdot 2 \cdot 2 \cdot 3 \cdot 3 \cdot 3 = 2^3 \cdot 3^3\). You can generalize this result as \((a \cdot b)^n = a^n \cdot b^n\).

**Step 10**  Consider the expression \(\frac{a^3}{a^3} \cdot \frac{a^3}{a^3} \cdot \frac{a^3}{a^3}\). By division, \(\frac{a^3}{a^3} = \frac{1}{a^3} \cdot \frac{a^3}{a^3} = 1\). By the property from Step 4, \(a^{3-3} = a^0\). Therefore, \(a^0 = 1\).

The properties of exponents are summarized on page 260 of your book. Be sure to read the power of a quotient property, power property of equality, and common base property of equality. These properties were not covered in the investigation. Try to come up with examples to convince yourself these properties are true.

Example A in your book illustrates a method for solving equations when both sides can be written as exponential expressions with a common base. Try to solve each equation before reading the solution. Here is another example:

**EXAMPLE A**  Solve.

a. \(125^x = 5\) \hspace{1cm} b. \(16^x = \frac{1}{64}\)

**Solution**  Convert each side of the equation to a common base, then use the properties of exponents.

a. \(125^x = 5\)  
   \((5^3)^x = 5^1\)  
   \(5^3 = 125\) and \(5^1 = 5\).  
   \(5^{3x} = 5^1\)  
   Use the power of a power property.  
   \(3x = 1\)  
   Use the common base property of equality.  
   \(x = \frac{1}{3}\)  
   Divide.

b. \(16^x = \frac{1}{64}\)  
   \((2^4)^x = \frac{1}{2^6}\)  
   \(2^4 = 16\) and \(2^6 = 64\).  
   \(2^{4x} = 2^{-6}\)  
   Use the power of a power property and the definition of negative exponents.  
   \(4x = -6\)  
   Use the common base property of equality.  
   \(x = -\frac{3}{2}\)  
   Divide.

An exponential function has a variable in the exponent. A **power function** has a variable in the base.

<table>
<thead>
<tr>
<th>Exponential function</th>
<th>Power function</th>
</tr>
</thead>
<tbody>
<tr>
<td>(y = ab^x), where (a) and (b) are constants</td>
<td>(y = ax^n), where (a) and (n) are constants</td>
</tr>
</tbody>
</table>

(continued)
Lesson 5.2 • Properties of Exponents and Power Functions (continued)

Example B in your book shows how you can use the properties of exponents to solve some equations involving variables raised to a power. Try to solve each equation yourself before reading the solution. Then, read the example below.

**EXAMPLE B**

Solve.

a. \(8x^3 = 4913\) \hspace{1cm} b. \(x^{4.5} = 512\)

**Solution**

Use the power property of equality and the power of a power property. Choose an exponent that will undo the exponent on \(x\).

a. \(8x^3 = 4913\) \hspace{1cm} Original equation.

\[x^3 = 614.125\] Divide both sides by 8.

\[(x^3)^{1/3} = 614.125^{1/3}\] Use the power property of equality.

\[x = 1.5\] Use your calculator to find \(614.125^{1/3}\).

b. \(x^{4.8} = 706\) \hspace{1cm} Original equation.

\[(x^{4.8})^{1/4.8} = 706^{1/4.8}\] Use the power property of equality.

\[x \approx 4\] Use your calculator to find \(706^{1/4.8}\).

In this book, the properties of exponents are defined only for positive bases. So, using these properties gives only one solution to each equation.
In this lesson you will

- learn how rational exponents are related to roots
- write equations for exponential curves in point-ratio form

You know that you can think of positive integer exponents as representing repeated multiplication. For example, \( 5^3 = 5 \cdot 5 \cdot 5 \). But how can you think about rational exponents? You will explore this question in the investigation.

Investigation: Getting to the Root

Complete the investigation in your book on your own. Then, compare your results with those below.

**Step 1** A calculator table shows that \( x^{1/2} \) is undefined for integers less than 0 and is a positive integer for values that are perfect squares. In fact, it appears that \( x^{1/2} = \sqrt{x} \).

<table>
<thead>
<tr>
<th>x</th>
<th>( x^{1/2} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>4</td>
<td>2</td>
</tr>
<tr>
<td>9</td>
<td>3</td>
</tr>
<tr>
<td>16</td>
<td>4</td>
</tr>
</tbody>
</table>

**Step 2** From the graph, it appears that \( y = x^{1/2} \) is equivalent to \( y = \sqrt{x} \). Graphing both functions in the same window verifies this.

**Step 3** Raising a number to a power of \( \frac{1}{2} \) is the same as taking its square root. For example, \( \sqrt{4} = 2 \) and \( 4^{1/2} = 2 \).

**Step 4** Here is a table for \( y = 25^x \), with \( x \) in increments of \( \frac{1}{2} \).

<table>
<thead>
<tr>
<th>x</th>
<th>( 25^x )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5</td>
<td>5</td>
</tr>
<tr>
<td>1</td>
<td>31.25</td>
</tr>
<tr>
<td>1.5</td>
<td>125</td>
</tr>
<tr>
<td>2</td>
<td>625</td>
</tr>
<tr>
<td>2.5</td>
<td>3125</td>
</tr>
</tbody>
</table>

**Step 5** Each entry in the table is the square root of 25, which is 5, raised to the numerator. That is, \( 25^{1/2} = 5^1 \), \( 25^{2/2} = 5^2 \), \( 25^{3/2} = 5^3 \), and so on. If this same pattern holds for \( y = 49^x \), then \( 49^{3/2} = (\sqrt{49})^3 = 7^3 = 343 \).

**Step 6** \( 27^{2/3} \) is the cube root of 27 raised to the second power:
\[
27^{2/3} = (\sqrt[3]{27})^2 = 3^2 = 9.
\]
Similarly, \( 85^{5/3} = (\sqrt[3]{85})^5 = 2^5 = 32 \).

**Step 7** To find \( a^{m/n} \), take the \( n \)th root of \( a \), and then raise the result to the power of \( m \). That is, \( a^{m/n} = (\sqrt[n]{a})^m \).

(continued)
Lesson 5.3 • Rational Exponents and Roots (continued)

In your book, read the paragraph following the investigation and the “Definition of Rational Exponents” box, which summarize what you discovered in the investigation. Then, try to solve the equations in Example A, which involve rational exponents.

Note that because you can write a function such as $y = \sqrt[7]{x^2}$ as $y = x^{3/7}$, it is considered to be a power function. You can apply all the transformations you learned in Chapter 4 to power functions. For example, the graph of $y = x^{3/7} + 3$ is the graph of $y = x^{3/7}$ shifted up 3 units.

Now, let’s turn our focus to exponential equations. In the general form of an exponential equation, $y = ab^x$, $b$ is the growth or decay factor. When you substitute 0 for $x$, you get $y = a$. This means that $a$ is the initial value of the function at time 0 (the $y$-intercept).

There are other useful forms of exponential equations. Recall that if you know a point $(x_1, y_1)$ on a line and the slope of the line, you can write an equation in point-slope form: $y = y_1 + m(x - x_1)$. Similarly, if you know a point $(x_1, y_1)$ on an exponential curve and the common ratio $b$ between points that are 1 horizontal unit apart (that is, the growth or decay factor), you can write an equation in point-ratio form: $y = y_1 \cdot b^{x-x_1}$.

Let’s look at an example to see how the general and point-ratio forms are related. Make a graph of the general equation $y = 47(0.9)^x$ on your calculator. Then, find the coordinates of a point on the curve. We’ll choose $(2, 38.07)$.

Using $(2, 38.07)$, you can write an equation in point-ratio form: $y = 38.07(0.9)^{x-2}$. A little algebra shows this is equivalent to the general equation:

$$y = 38.07(0.9)^{x-2} \quad \text{Point-ratio form.}$$

$$= 38.07(0.9)^x(0.9)^{-2} \quad \text{Use the multiplication property of exponents.}$$

$$= 47(0.9)^x \quad \text{Use your calculator to multiply 38.07 \cdot (0.9)^{-2}.}$$

Work through Example B in your book. Pay special attention to the technique used to find the value of $b$.

The text after Example B explains another way to find the equation without having to solve for $b$. Read that text very carefully. Here is a summary of the method:

1. From part a in Example B, the graph passes through $(4, 40)$ and $(7.2, 4.7)$. The ratio of $y$-values for the two points is $\frac{4.7}{40}$. Note that this is not the value of $b$ because the $x$-values 4 and 7.2 are not consecutive integers; the ratio $\frac{4.7}{40}$ is spread over 3.2 horizontal units, rather than over 1 unit.

2. Write the equation of the curve that passes through $(4, 40)$ and that does have a $b$-value of $\frac{4.7}{40}$: $y = 40\left(\frac{4.7}{40}\right)^{x-4}$.

3. Start with the equation from the previous step. Divide the $x$-value by 3.2 to stretch the graph horizontally by 3.2 units: $y = 40\left(\frac{4.7}{40}\right)^{\frac{x-4}{3.2}}$. 

64   CHAPTER 5   Discovering Advanced Algebra Condensed Lessons   ©2010 Kell Hu Pulisi
In this lesson you will

- solve application problems involving exponential and power functions

The examples in your book are applications of exponential and power functions. In both examples, the problem is solved by writing an equation and then undoing the order of operations.

Work through Example A in your book. Then read the text below it and make sure you understand the term asymptote. Read the example below. Try to solve the problem yourself before reading the solution.

**EXAMPLE A**  
Computer Central is going out of business in 10 weeks. The owner wants to lower the price of every item in the store by the same percentage each week so that by the end of 10 weeks any remaining merchandise is on sale for 25% of its original price. By what percentage does he need to lower the prices each week in order for this to happen?

**Solution**  
By the end of 10 weeks, the price of a computer originally marked $1,000 should be reduced to $250. The discount rate, \( r \), is unknown. Write an equation and solve for \( r \).

\[
250 = 1000(1 - r)^{10} \quad \text{Original equation.}
\]
\[
0.25 = (1 - r)^{10} \quad \text{Undo the multiplication by 1000 by dividing both sides by 1000.}
\]
\[
0.25^{1/10} = \left( (1 - r)^{10} \right)^{1/10} \quad \text{Undo the power of 10 by raising both sides to the power of \( \frac{1}{10} \).}
\]
\[
0.25^{1/10} = 1 - r \quad \text{Use the properties of exponents.}
\]
\[
r = 1 - 0.25^{1/10} \quad \text{Add \( r \) and \(-0.25^{1/10}\) to both sides.}
\]
\[
r = 0.1294 \quad \text{Use a calculator to evaluate \( 1 - 0.25^{1/10} \).}
\]

The owner will need to reduce the prices by about 13% each week.

Notice that it doesn’t matter what original price you start with. For example, if you start with a price of $2,400, your equation would be 600 = 2,400\((1 - r)^{10}\). After dividing both sides by 2,400, you would have the same equation in the second step above.

Example B in your book is an application of the point-ratio form of an exponential equation. This problem has a twist, though, because the exponential function must be translated so that it approaches a nonzero long-run value. Work through the example with a pencil and paper. Then, work through the example on the following page.

(continued)
EXAMPLE B

This table shows the amount of chlorine in a swimming pool every fourth day. Find an equation that models the data in the table.

<table>
<thead>
<tr>
<th>Day x</th>
<th>0</th>
<th>4</th>
<th>8</th>
<th>12</th>
<th>16</th>
<th>20</th>
<th>24</th>
</tr>
</thead>
<tbody>
<tr>
<td>Chlorine (g) y</td>
<td>300</td>
<td>252.20</td>
<td>227.25</td>
<td>214.22</td>
<td>207.43</td>
<td>203.88</td>
<td>202.02</td>
</tr>
</tbody>
</table>

Solution

Plot the data. The graph shows a curved shape, so the data are not linear. The pattern appears to be a decreasing geometric sequence, so an exponential decay equation will provide the best model. However, notice that the long-run value appears to be 200, not 0.

The exponential decay function in point-ratio form is \( y = y_1 \cdot b^{x-x_1} \). However, because this function approaches a long-run value of 0, it must be translated up 200 units, to have a horizontal asymptote at \( y = 200 \).

To do this, replace \( y \) with \( y - 200 \). Because the coefficient, \( y_1 \), is also a \( y \)-value, it must be replaced with \( y_1 - 200 \) to account for the translation.

The point-ratio equation is now \( y - 200 = (y_1 - 200) \cdot b^{x-x_1} \).

To find the value of \( b \), substitute the value of any point, say (20, 203.88), for \( (x_1, y_1) \) and solve for \( b \).

\[
\begin{align*}
  y - 200 &= (y_1 - 200) \cdot b^{x-x_1} & \text{Original equation.} \\
  y - 200 &= (203.88 - 200) \cdot b^{x-20} & \text{Substitute (20, 203.88) for \( (x_1, y_1) \).} \\
  y - 200 &= 3.88 \cdot b^{x-20} & \text{Subtract within parentheses.} \\
  \frac{y - 200}{3.88} &= b^{x-20} & \text{Divide both sides by 3.88.} \\
  \left(\frac{y - 200}{3.88}\right)^{1/(x-20)} &= b & \text{Raise both sides to a power of } \frac{1}{x-20} \text{ to solve for } b.
\end{align*}
\]

You now have \( b \) in terms of \( x \) and \( y \). Evaluating \( b \) for all the other data points gives these values:

<table>
<thead>
<tr>
<th>Day x</th>
<th>0</th>
<th>4</th>
<th>8</th>
<th>12</th>
<th>16</th>
<th>20</th>
<th>24</th>
</tr>
</thead>
<tbody>
<tr>
<td>Chlorine (g) y</td>
<td>300</td>
<td>252.20</td>
<td>227.25</td>
<td>214.22</td>
<td>207.43</td>
<td>203.88</td>
<td>202.02</td>
</tr>
<tr>
<td>( b )</td>
<td>0.850</td>
<td>0.8501</td>
<td>0.8501</td>
<td>0.8501</td>
<td>0.8501</td>
<td>0.8494</td>
<td></td>
</tr>
</tbody>
</table>

All the values for \( b \) are close to 0.85, so use that value for \( b \). Therefore, a model for these data is \( y - 200 = (203.88 - 200) \cdot 0.85^{x-20} \), or \( y = 200 + 3.88(0.85)^{x-20} \).
In this lesson you will
- find inverses of functions
- learn how the graph and equation of a function are related to the graph and equation of its inverse
- compose a function with its inverse

Look at the two graphs at the beginning of Lesson 5.5 in your book. These graphs represent the same data. However, the independent variable in the left graph is the dependent variable in the right graph, and the dependent variable in the left graph is the independent variable in the right graph.

A relation that results from exchanging the independent and dependent variables of a function is called the inverse of the function.

Investigation: The Inverse

In this investigation you’ll discover how the equations for a function and its inverse are related. Complete the steps for the investigation on your own, and then compare your results with those below.

**Step 1** The graph and table for \( f(x) = 6 + 3x \) are shown at right.

**Step 2** To complete the table for the inverse, fill in the \( f(x) \)-values from Step 1 as the \( x \)-values.

<table>
<thead>
<tr>
<th>( x )</th>
<th>3</th>
<th>6</th>
<th>9</th>
<th>12</th>
<th>15</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y )</td>
<td>−1</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>3</td>
</tr>
</tbody>
</table>

**Step 3** The points lie on a line with slope \( \frac{1}{3} \) and \( y \)-intercept −2. The line with equation \( y = \frac{1}{3}x - 2 \) (or any equivalent equation) passes through the points.

**Step 4**

i. The table and graph of \( g(x) = \sqrt{x + 1} - 3 \) are shown at right.

To complete the table for the inverse of \( g(x) = \sqrt{x + 1} - 3 \), fill in the \( g(x) \)-values as the \( x \)-values.

<table>
<thead>
<tr>
<th>( x )</th>
<th>−3</th>
<th>−2</th>
<th>−1.59</th>
<th>−1.27</th>
<th>−1</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y )</td>
<td>−1</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>3</td>
</tr>
</tbody>
</table>

(continued)
Lesson 5.5 • Building Inverses of Functions (continued)

The graph of the inverse table values shows that the points appear to lie on a parabola with vertex \((-3, -1)\). The graph at right shows that the equation \(y = (x + 3)^2 - 1\) fits the points. Your equation should be equivalent.

**ii.** The table and graph of \(h(x) = (x - 2)^2 - 5\) are shown at right. To complete the table for the inverse of \(h(x) = (x - 2)^2 - 5\), fill in the \(h(x)\)-values as the \(x\)-values.

<table>
<thead>
<tr>
<th>(x)</th>
<th>4</th>
<th>-1</th>
<th>-4</th>
<th>-5</th>
<th>-4</th>
</tr>
</thead>
<tbody>
<tr>
<td>(y)</td>
<td>-1</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>3</td>
</tr>
</tbody>
</table>

When you graph the inverse table values, the graph appears to show a sideways parabola. Because points \((-4, 1)\) and \((-4, 3)\) have the same \(x\)-value, but two different \(y\)-values, this graph does not represent a function. You will need to use two equations to describe this graph.

The bottom half of the graph can be described by a square root equation that is reflected vertically. The vertex is at \((-5, 2)\), so the equation is \(y = -\sqrt{x + 5} + 2\). To graph the upper half, use the equation \(y = \sqrt{x + 5} + 2\). The graphs fit the points, as you can see at right.

**Step 5** The graph of each inverse is a reflection of the original function over the line \(y = x\). Look at the graphs of \(f(x) = 6 + 3x\) and its inverse \(y = \frac{1}{3}x - 2\) at right. Imagine folding the coordinate plane along the line \(y = x\). The graphs would match up exactly. You might try this on graph paper for each pair of graphs from Step 4.

**Step 6** Start with the original function, \(f(x) = 6 + 3x\). Switch the independent and dependent variables to get \(x = 6 + 3y\). Now solve this equation for \(y\). You should get \(y = \frac{x - 6}{3}\). You can verify with a graph or by symbolic methods that this is equivalent to the equation \(y = \frac{1}{3}x - 2\). Try this method for the equations from Step 4. You should find that switching the \(x\) and \(y\)-variables and solving for \(y\) gives you the inverse equation for the original equation.

_____

Solve the problem stated in Example A in your book, and then read the solution.

In the investigation you may have noticed that the inverse of a function may not be a function. For example, the inverse of the function \(y = (x - 2)^2 - 5\) is \(y = \pm\sqrt{x + 5} + 2\), which pairs every \(x\)-value (except \(-5\)) with two \(y\)-values.

(continued)
Lesson 5.5 • Building Inverses of Functions (continued)

When a function and its inverse are both functions, the function is called a one-to-one function (because there is a one-to-one correspondence between the domain values and the range values). A function is one-to-one if its graph passes both the vertical line test and the horizontal line test. The inverse of a one-to-one function \( f(x) \) is written as \( f^{-1}(x) \).

Example B in your book illustrates that when you compose a function with its inverse, you get \( x \). The example below uses a different function to illustrate the same point.

**EXAMPLE**

Consider the function \( f(x) = \sqrt{x + 1} - 3 \). Find \( f(f^{-1}(x)) \) and \( f^{-1}(f(x)) \).

**Solution**

In the investigation you found that \( f^{-1}(x) = (x + 3)^2 - 1 \). First, find \( f(f^{-1}(x)) \).

Because \( f(x) \) has the range \( y \geq -3 \), you must restrict the domain for \( f^{-1}(x) \) to \( x \geq -3 \).

\[
\begin{align*}
  f(f^{-1}(x)) &= \sqrt{(x + 3)^2 - 1} + 1 - 3 \\
  &= \sqrt{x^2 + 6x + 9} - 3 \\
  &= \sqrt{(x + 3)^2} - 3 \\
  &= x + 3 - 3 \\
  &= x
\end{align*}
\]

Substitute \( f^{-1}(x) \) for \( x \).

Expand \((x + 3)^2\) and simplify the expression under the square root sign.

Factor the expression under the square root sign.

Because \( x \geq -3 \), \( \sqrt{(x + 3)^2} = x + 3 \).

Add.

Now, find \( f^{-1}(f(x)) \).

\[
\begin{align*}
  f^{-1}(f(x)) &= (\sqrt{x + 1} - 3 + 3)^2 - 1 \\
  &= (\sqrt{x + 1})^2 - 1 \\
  &= x + 1 - 1 \\
  &= x
\end{align*}
\]

Substitute \( f(x) \) for \( x \).

Subtract.

\((\sqrt{x + 1})^2 = x + 1\)

Subtract.
In this lesson you will

- learn the meaning of a logarithm
- use logarithms to solve exponential equations

In Lesson 5.2, you solved some equations in which \( x \) was an exponent. In all the equations, you were able to write both sides with a common base. Unfortunately, this is usually not possible. In this lesson, you will discover a powerful method for solving for \( x \) in an exponential equation. This method involves a new function called a logarithm, abbreviated log.

**Investigation: Exponents and Logarithms**

In this investigation you will explore the connection between exponents on the base 10 and logarithms. Work through the steps of the investigation on your own. Then compare your results with the ones below.

**Step 1** The graph of \( f(x) = 10^x \) is shown below, along with information about the function.

<table>
<thead>
<tr>
<th>( x )</th>
<th>( f(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( -1.5 )</td>
<td>0.032</td>
</tr>
<tr>
<td>( -1 )</td>
<td>0.1</td>
</tr>
<tr>
<td>( -0.5 )</td>
<td>0.316</td>
</tr>
<tr>
<td>( 0 )</td>
<td>1</td>
</tr>
<tr>
<td>( 0.5 )</td>
<td>3.162</td>
</tr>
<tr>
<td>( 1 )</td>
<td>10</td>
</tr>
<tr>
<td>( 1.5 )</td>
<td>31.62</td>
</tr>
</tbody>
</table>

**Step 2** Remember that once you have completed the output values for the original function, you can fill those in as the input values for the inverse. The tables are completed below.

| \( x \) | \(-1.5\) | \(-1\) | \(-0.5\) | 0 | 0.5 | 1 | 1.5 |
| \( f^{-1}(x) \) | 0.032 | 0.1 | 0.316 | 1 | 3.162 | 10 | 31.62 |

**Step 3** Plot the points in the inverse table as a scatter plot. Adjust your window so that all seven points are visible. The graph is shown below, along with its table of information.

| \( x \) | \( x > 0 \) |
| \( y \)-intercept | none |
| \( y \)-intercept | 1 |

(continued)
Lesson 5.6 • Logarithmic Functions (continued)

Step 4  This inverse is called the logarithm of \(x\), or \(\log(x)\). The answers for 4a–h are shown below. Make sure that you are able to find these values using your calculator. See Calculator Note 5C to learn how to work with logarithms on your calculator.

\[
\begin{align*}
a. \quad 10^{1.5} &\approx 31.6 & \text{b.} \quad \log(10^{1.5}) &\approx 1.5 & \text{c.} \quad \log 0.32 &\approx -0.49 & \text{d.} \quad 10^{\log 0.32} &\approx 0.32 \\
e. \quad 10^{1.2} &\approx 15.8 & \text{f.} \quad \log(10^{1.2}) &\approx 1.2 & \text{g.} \quad 10^{\log 25} &\approx 25 & \text{h.} \quad \log 10^{2.8} &\approx 2.8
\end{align*}
\]

Step 5  Examine the answers for b, f, and h. It appears that the logarithm gives the exponent on 10. So \(\log 10^x = x\).

Step 6  Look at the answers for d and g. The logarithm seems to “undo” the exponent. So, \(10^{\log x} = x\).

Step 7  If you weren’t able to complete these statements on your own, try them now before checking against the results below. You can verify these answers with your calculator.

\[
\begin{align*}
a. \quad \text{If } 100 = 10^2, \text{ then } \log 100 &\approx 2. \\
b. \quad \text{If } 400 = 10^{2.6021}, \text{ then } \log 400 &\approx 2.6021. \\
c. \quad \text{If } 500 = 10^{2.6990}, \text{ then } \log 500 &\approx 2.6990.
\end{align*}
\]

Step 8  If \(y = 10^x\), then \(\log y = x\).

Log \(x\) is the exponent you put on the base 10 to get \(x\). For example, \(\log 1000 = 3\) because \(10^3 = 1000\). You can find logarithms for other bases as well. The base is specified as a subscript after the word “log.” For example, \(\log_{10} 32\) is 5, the exponent you have to put on 2 to get 32. If no base is specified, \(\log x\) is assumed to be the base-10 logarithm (that is, \(\log x\) means \(\log_{10} x\)). Logarithms with base 10 are called common logarithms.

Solve the equation given in Example A in your book, and then read the solution. Then, read the short paragraph after Example A and the definition of logarithm.

Example B shows how to use logarithms to solve an exponential equation when the base is not 10. Read Example B carefully and make sure you understand each step of the solution. Check your understanding by solving the equation in the following example.

**EXAMPLE**  Solve \(7^x = 211\).

\[\textbf{Solution}\]

Rewrite each side of the equation as a power with a base of 10.

\[
\begin{align*}
7^x &= 211 & \text{Original equation.} \\
(10^{\log 7})^x &= 10^{\log 211} & \text{Use the fact that } a = 10^{\log_a}.
\end{align*}
\]

\[
\log 7 \cdot x &= \log 211 & \text{Use the common base property of equality.}
\]

\[
x &= \frac{\log 211}{\log 7} & \text{Divide both sides by } \log 7.
\]

\[
x \approx 2.7503 & \text{ Use a calculator to evaluate.}
\]

Look at the original equation, \(7^x = 211\), in the example above. The solution to this equation is the exponent you have to put on 7 to get 211. In other words, \(x = \log_7 211\). The fourth step of the solution indicates that \(\log 211 = \frac{\log_{10} 211}{\log_{10} 7}\). This illustrates the logarithm change-of-base property. Read this property in your book. Then, work through Example C in your book.
In this lesson you will
- use logarithms to help you do complex calculations
- explore the properties of logarithms

Read the first paragraph of the lesson on page 293 of your book. Then, read the example, following along with a pencil and paper. The solutions to all three parts use the fact that \( m = 10^{\log m} \). The example below will give you more practice. Solve each part on your own before reading the solution.

**EXAMPLE**

Convert numbers to logarithms to do these problems.

a. Find \( 37.678 \cdot 127.75 \) without using the multiplication key on your calculator.

b. Find \( \frac{37.678}{127.75} \) without using the division key on your calculator.

c. Find \( 9.3^{1.8} \) without using the exponentiation key on your calculator.

**Solution**

a. \( 37.678 \cdot 127.75 = 10^{\log 37.678} \cdot 10^{\log 127.75} = 10^{\log 37.678 + \log 127.75} = 4813.3645 \)

b. \( \frac{37.678}{127.75} = \frac{10^{\log 37.678}}{10^{\log 127.75}} = 10^{\log 37.678 - \log 127.75} = 0.2949 \)

c. \( (10^{\log 9.3})^{1.3} = 10^{(\log 9.3) \cdot 1.3} \approx 18.1563 \)

Before calculators, people did computations such as those above using tables of base-10 logarithms. For example, to find \( 37.678 \cdot 127.75 \), they looked up \( \log 37.678 \) and \( \log 127.75 \) and added those numbers. Then, they worked backward to find the antilog, or antilogarithm, of the sum. The antilog of a number is 10 raised to that number. For example, the antilog of 3 is \( 10^3 \), or 1000.

**Investigation: Properties of Logarithms**

In this investigation you will explore the properties of logarithms. Complete the investigation. Then, compare your results with those below.

**Step 1** The completed table is shown at right.

**Step 2** Here are six sample answers. You may find others.

\[
\begin{align*}
\log 2 + \log 3 &= \log 6 \\
\log 2 + \log 5 &= \log 10 \\
\log 2 + \log 6 &= \log 12 \\
\log 2 + \log 8 &= \log 16 \\
\log 3 + \log 5 &= \log 15 \\
\log 3 + \log 9 &= \log 27
\end{align*}
\]

**Step 3** The sum of \( \log a \) and \( \log b \) is equal to the log of the product of \( a \) and \( b \).

**Step 4** Here are three possible answers. You may find others.

\[
\begin{align*}
\log 9 + \log 10 &= \log 90 \\
\log 3 + \log 10 &= \log 30 \\
\log 8 + \log 9 &= \log 72
\end{align*}
\]

You can represent the pattern with the equation: \( \log a + \log b = \log ab \).
Lesson 5.7 • Properties of Logarithms (continued)

**Step 5**  Here are six sample answers. There are many others. Try to find a few more.

\[
\begin{align*}
\log 6 - \log 2 &= \log 3 & \log 6 - \log 3 &= \log 2 & \log 10 - \log 2 &= \log 5 \\
\log 10 - \log 5 &= \log 2 & \log 12 - \log 2 &= \log 6 & \log 15 - \log 3 &= \log 5
\end{align*}
\]

You can represent the pattern with the equation: \( \log a - \log b = \log \left( \frac{a}{b} \right) \).

**Step 6**  Here are four possible answers. You may find others.

\[
\begin{align*}
3 \log 2 &= \log 8 & 3 \log 3 &= \log 27 & 4 \log 2 &= \log 16 & 2 \log 5 &= \log 25
\end{align*}
\]

You can represent the pattern with the equation \( b \log a = \log a^b \).

**Step 7**  Because logarithms are exponents, they have properties similar to the properties of exponents that you studied earlier. For example, the property you discovered about the sum of logs, \( \log a + \log b = \log ab \), is related to the product property of exponents: \( a^m \cdot a^n = a^{m+n} \). What other connections can you find between the properties of logarithms and the properties of exponents?

The properties of exponents and logarithms are summarized on page 296 of your book. Read them carefully.
Applications of Logarithms

In this lesson you will

- use logarithms to solve real-world problems that can be modeled with exponential equations
- use a technique called curve straightening to determine whether a relationship is exponential

In this lesson you will use what you have learned about logarithms and their properties to solve some problems. First, work through Example A in your book. The fifth step of that solution involves taking the logarithm of both sides of the equation. It is important to remember that you can take the logarithm of both sides only if you know the value of each side is positive. (The logarithm function is not defined for zero or negative numbers.)

The example below shows you how to solve Exercise 5c in your book. Try to solve the problem yourself before looking at the solution.

**EXAMPLE**

The equation \( f(x) = \frac{12,000}{1 + 499(1.09)^{-x}} \) gives the total sales \( x \) days after the release of a new video game. On which day were 6,000 games sold?

**Solution**

Substitute 6000 for \( f(x) \) and solve.

\[
6,000 = \frac{12,000}{1 + 499(1.09)^{-x}}
\]

Multiply both sides by \( 1 + 499(1.09)^{-x} \).

\[
6,000[1 + 499(1.09)^{-x}] = 12,000
\]

Divide both sides by 6,000 and then subtract 1 from both sides.

\[
499(1.09^{-x}) = 1
\]

Divide both sides by 499.

\[
1.09^{-x} = \frac{1}{499}
\]

Take the logarithm of both sides.

\[
-x \cdot \log(1.09) = \log\left(\frac{1}{499}\right)
\]

Multiply both sides by \(-\frac{1}{\log(1.09)}\).

\[
x = 72.1
\]

Evaluate using a calculator.

Six thousand games were sold 72 days after the release.

Example B in your book illustrates a technique called curve straightening. Work through this example, following along with a pencil and paper. Curve straightening is also used in the investigation.
Lesson 5.8 • Applications of Logarithms (continued)

Investigation: Cooling

**Step 1** Read Step 1 of the investigation in your book. If you have access to a temperature probe, you can collect your own data. If not, use the sample data below.

**Step 2** Let \( t \) be time in seconds and let \( p \) be the temperature of the probe in °C. Sketch a graph of what you think the \((t, p)\) data will look like.

**Step 3** Complete Step 3 in your book. Below are a table and graph of some sample data.

<table>
<thead>
<tr>
<th>Time (s)</th>
<th>Temperature (°C)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>29.7495</td>
</tr>
<tr>
<td>10</td>
<td>28.87</td>
</tr>
<tr>
<td>20</td>
<td>27.4995</td>
</tr>
<tr>
<td>30</td>
<td>26.4995</td>
</tr>
<tr>
<td>40</td>
<td>25.812</td>
</tr>
<tr>
<td>50</td>
<td>25.312</td>
</tr>
<tr>
<td>60</td>
<td>24.8745</td>
</tr>
<tr>
<td>70</td>
<td>24.437</td>
</tr>
<tr>
<td>80</td>
<td>24.062</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Time (s)</th>
<th>Temperature (°C)</th>
</tr>
</thead>
<tbody>
<tr>
<td>90</td>
<td>23.8745</td>
</tr>
<tr>
<td>100</td>
<td>23.6245</td>
</tr>
<tr>
<td>110</td>
<td>23.3745</td>
</tr>
<tr>
<td>120</td>
<td>23.187</td>
</tr>
<tr>
<td>130</td>
<td>22.9995</td>
</tr>
<tr>
<td>140</td>
<td>22.812</td>
</tr>
<tr>
<td>150</td>
<td>22.687</td>
</tr>
<tr>
<td>160</td>
<td>22.562</td>
</tr>
<tr>
<td>170</td>
<td>22.437</td>
</tr>
<tr>
<td>180</td>
<td>22.312</td>
</tr>
</tbody>
</table>

The plot shows exponential decay, and the limit appears to be 22°. If 22° is the limit, then an equation in the form \( p = 22 + ab^t \), or \( p - 22 = ab^t \), will model the data. Taking the log of both sides gives \( \log(p - 22) = \log a + t \cdot \log b \), which is a linear equation. So, if the limit is 22°, then the graph of \( \log(p - 22) \) will be linear.

**Step 4** Subtract 22° from each temperature, and plot \((t, \log(p - 22))\). If you are using your own data, you might delete any repeated values at the end of your data set. Graphing the sample data gives the plot below, which appears to be linear. Therefore, 22° is the limit.
Lesson 5.8 • Applications of Logarithms (continued)

Step 5  Now you need to find an equation that models the data. Use the median-median line.

\[
\hat{y} = -0.007x + 0.88
\]

Median-median line for \((t, \log(p - 22))\) data.

\[
\log(p - 22) = -0.007t + 0.88
\]

Substitute \(\log(p - 22)\) for \(y\) and \(t\) for \(x\).

\[
p - 22 = 10^{-0.007t + 0.88}
\]

Definition of logarithm.

\[
p = 10^{-0.007t + 0.88} + 22
\]

Add 22 to both sides.

\[
p = 10^{-0.007t} \cdot 10^{0.88} + 22
\]

Multiplication property of exponents.

\[
p = 10^{-0.007t} \cdot 7.58 + 22
\]

Evaluate \(10^{0.88}\).

\[
p = (10^{-0.007})^t \cdot 7.58 + 22
\]

Power property of exponents.

\[
p = 0.98^t \cdot 7.58 + 22
\]

Evaluate \(10^{-0.007}\).

\[
\hat{y} = 22 + 7.58(0.98)^t
\]

Rewrite in the form \(y = k + ab^x\).

You can use a calculator to check the fit.