In this lesson you will

- write explicit formulas for arithmetic sequences
- write linear equations in intercept form

You learned about recursive formulas in Chapter 1. Using a recursive formula to find a term far along in a sequence can be tedious. For example, to find the value of $u_{72}$, you first have to find the values of $u_1$ through $u_{71}$. An explicit formula tells you how to calculate any term of the sequence without calculating the previous terms. The recursive formula and explicit formula below represent the same sequence.

**Recursive formula**

$$u_0 = 5$$

$$u_n = u_{n-1} + 7 \text{ where } n \geq 1$$

**Explicit formula**

$$u_n = 5 + 7n$$

Use both formulas to calculate the first few terms of the sequence. Do you get the same results? To find the value of $u_{72}$ using the explicit formula, substitute 72 for $n$:

$$u_{72} = 5 + 7(72) = 509$$

To learn more about explicit formulas, read the text through Example A in your book. Then, work through the example below.

**EXAMPLE**

Consider the recursively defined arithmetic sequence

$$u_0 = 13$$

$$u_n = u_{n-1} - 3 \text{ where } n \geq 1$$

**a.** Find an explicit formula for the sequence.

**b.** Use the explicit formula to find $u_{17}$.

**c.** Find the value of $n$ so that $u_n = -50$.

**Solution**

**a.** To generate the terms, you start with 13 and subtract another 3 for each term:

$$u_0 = 13$$

$$u_1 = 10 = 13 - 3 = 13 - 3 \cdot 1$$

$$u_2 = 7 = 13 - 3 - 3 = 13 - 3 \cdot 2$$

$$u_3 = 4 = 13 - 3 - 3 - 3 = 13 - 3 \cdot 3$$

Each term is equal to 13 minus 3 times the term number. So, the explicit formula for the $n$th term is

$$u_n = 13 - 3n$$

**b.** Start with the explicit formula and substitute 17 for $n$.

$$u_{17} = 13 - 3(17) = -38$$

(continued)
Lesson 3.1 • Linear Equations and Arithmetic Sequences (continued)

c. Substitute \(-50\) for \(u_n\) in the explicit formula and solve for \(n\).

\[
\begin{align*}
-50 &= 13 - 3n & \text{Substitute } -50 \text{ for } u_n, \\
-63 &= -3n & \text{Subtract } 13 \text{ from both sides.} \\
n &= 21 & \text{Divide both sides by } -3.
\end{align*}
\]

The variable \(n\) in the explicit formula \(u_n = 13 - 3n\) stands for a whole number. So, if you graph the sequence of ordered pairs \((n, u_n)\), you get a set of discrete points.

The points lie on a line with a slope equal to \(-3\), the common difference of the arithmetic sequence. The point \((0, 13)\), which corresponds to the starting term of the sequence, is the \(y\)-intercept of the line. So, the equation for the line through the points is \(y = 13 - 3x\), or \(y = -3x + 13\).

In this course, you will use \(x\) and \(y\) to write linear equations and \(n\) and \(u_n\) to write recursive and explicit formulas for sequences of discrete points.

Investigation: Match Point

Step 1 The investigation in your book gives three recursive formulas, three graphs, and three linear equations. Match the formulas, graphs, and equations that go together. If a formula, graph, or equation is missing, you will need to create it. When you are finished, read the answers and explanations below.

1, B, \(y = 4 - x\): The sequence with Formula 1 has starting value 4 and constant difference \(-1\). The graph should therefore have a point at \((0, 4)\), and then each subsequent point should be 1 unit lower than the previous point. Graph B fits this description. The starting value, 4, is the \(y\)-intercept of the line through the points, and the constant difference, \(-1\), is the slope. So, the linear equation is \(y = 4 - x\).

2, C, iii: The sequence with Formula 2 has starting value 2 and constant difference 5. The graph should therefore have a point at \((0, 2)\), and then each subsequent point should be 5 units higher than the previous point. Graph C fits this description. The starting value, 2, is the \(y\)-intercept of the line through the points, and the constant difference, 5, is the slope. So, the linear equation is \(y = 2 + 5x\), which is Equation iii.

3, see graph at right, i: The sequence with Formula 3 has starting value \(-4\) and constant difference 3. The graph should therefore have a point at \((0, -4)\), and then each subsequent point should be 3 units higher than the previous point. This is shown in the graph at right. The starting value, \(-4\), is the \(y\)-intercept of the line through the points, and the constant difference, 3, is the slope. So, the linear equation is \(y = -4 + 3x\), which is Equation i.

A, \(u_0 = 3\) and \(u_n = u_{n-1} + 2\) where \(n \geq 1\), \(y = 3 + 2x\): Graph A has a point at \((0, 3)\), and then each subsequent point is 2 units higher than the previous point, so the sequence corresponding to (continued)
Lesson 3.1 • Linear Equations and Arithmetic Sequences (continued)

Graph A has starting value 3 and constant difference 2. This sequence has recursive formula \( u_0 = 3 \) and \( u_n = u_{n-1} + 2 \) where \( n \geq 1 \). The line through the points in Graph A has slope 2 and \( y \)-intercept 3, so it has equation \( y = 3 + 2x \).

ii, \( u_0 = 4 \) and \( u_n = u_{n-1} + 1 \) where \( n \geq 1 \), see graph at right: The sequence corresponding to Equation ii has starting value 4 and constant difference 1, so its formula is \( u_0 = 4 \) and \( u_{n-1} + 1 \) where \( n \geq 1 \). The graph of the sequence has a point at (0, 4), and then each subsequent point is 1 unit higher than the previous point. This is shown in the graph at right.

**Step 2** The starting value of an arithmetic sequence is the \( y \)-intercept of the line through the points and the value of \( a \) in the line’s equation, \( y = a + bx \). The common difference of an arithmetic sequence is the slope of the line through the points and the value of \( b \) in the line’s equation, \( y = a + bx \).

**Step 3** The points \((n, u_n)\) of an arithmetic sequence are always collinear because to get from one point to the next, you always move over 1 unit and up \( b \) units, where \( b \) is the constant difference. Therefore, the slope between any two points is \( b \), so they must lie on the same line.

Example B in your book gives you more practice working with explicit formulas and linear equations. Work through the example on your own and then read the rest of the lesson.
In this lesson you will

- use the slope formula
- conduct an experiment and fit a line to the data
- identify the dependent variable, independent variable, domain, and range of a relationship

In previous math classes, you learned that the formula for the slope of the line between two points \((x_1, y_1)\) and \((x_2, y_2)\) is

\[
slope = \frac{y_2 - y_1}{x_2 - x_1}
\]

where \(x_1 \neq x_2\).

For any two points on the same line, you will get the same slope. In other words, a line has only one slope.

Horizontal lines are the only lines that have two points with the same \(y\)-value. (In fact, every point on a horizontal line has the same \(y\)-value.) You can see from the formula that the slope of a horizontal line is 0.

Vertical lines are the only lines that have two points with the same \(x\)-value. (In fact, every point on a vertical line has the same \(x\)-value.) The slope of a vertical line is undefined because the denominator in the slope formula is 0.

As you know, when a linear equation is written in intercept form, \(y = a + bx\), the slope of the line is \(b\), the coefficient of \(x\). Many books use the letter \(m\) to represent the slope, but we will use the letter \(b\).

When real-world data show a linear trend, you can fit a line to the data. Unless the data are exactly linear, the slope of the line will depend on the points you choose to draw the line through.

When you analyze the relationship between two variables, you need to decide which variable to express in terms of the other. When one variable depends on another variable, it is called the dependent variable. The other variable is called the independent variable. You also need to think about the domain and range of the relationship. The domain is the set of possible \(x\)-values, and the range is the set of possible \(y\)-values.

**Investigation: Balloon Blastoff**

In this investigation, you will write an equation for the distance of a balloon rocket from a sensor as a function of time.

Read the Procedure Note and Steps 1 and 2 in your book. If you have the materials to conduct the experiment and a friend who can help you, collect your own data. Otherwise, use the sample data on the next page. Complete the steps on your own before reading the results on the next page.

(continued)
Lesson 3.2 • Revisiting Slope (continued)

<table>
<thead>
<tr>
<th>Time (s)</th>
<th>Distance (m)</th>
</tr>
</thead>
<tbody>
<tr>
<td>x</td>
<td>y</td>
</tr>
<tr>
<td>0</td>
<td>0.132667</td>
</tr>
<tr>
<td>0.05</td>
<td>0.132099</td>
</tr>
<tr>
<td>0.1</td>
<td>0.191364</td>
</tr>
<tr>
<td>0.15</td>
<td>0.276836</td>
</tr>
<tr>
<td>0.2</td>
<td>0.368507</td>
</tr>
<tr>
<td>0.25</td>
<td>0.473591</td>
</tr>
<tr>
<td>0.3</td>
<td>0.664749</td>
</tr>
<tr>
<td>0.35</td>
<td>0.941310</td>
</tr>
<tr>
<td>0.4</td>
<td>1.091230</td>
</tr>
<tr>
<td>0.45</td>
<td>1.298230</td>
</tr>
<tr>
<td>0.5</td>
<td>1.518573</td>
</tr>
<tr>
<td>0.55</td>
<td>1.744151</td>
</tr>
<tr>
<td>0.6</td>
<td>1.999602</td>
</tr>
<tr>
<td>0.65</td>
<td>2.365544</td>
</tr>
<tr>
<td>0.7</td>
<td>2.394023</td>
</tr>
</tbody>
</table>

**Step 3** Here is a graph of the data, with time as the independent variable. The domain of the data is $0 \leq x \leq 0.7$, and the range is $0.132009 \leq y \leq 2.394023$. The domain indicates the number of seconds the rocket is in motion. The range indicates the distance it travels.

**Steps 4 and 5** We’ll use $A(0.05, 0.132099)$, $B(0.3, 0.664749)$, $C(0.5, 1.518573)$, and $D(0.65, 2.365544)$ as representative points.

- Slope between $A$ and $B$: 2.1306
- Slope between $A$ and $C$: 3.0812
- Slope between $A$ and $D$: 3.7224
- Slope between $B$ and $C$: 4.2691
- Slope between $B$ and $D$: 4.8594
- Slope between $C$ and $D$: 5.6464

**Step 6** The slope estimates are all different because the four points do not all lie on the same line. The mean of the slopes is 3.9515, and the median is 3.9958. There is no mode. The mean and median are very close. Either one would be a reasonable choice for the representative slope. We’ll use the mean.

**Step 7** The slope indicates that the distance from the rocket to the sensor increases by 3.9515 meters every second. In other words, the rocket’s speed is about 3.95 meters per second.

Now, work through the example in your book.
In this lesson you will

- draw a **line of fit** for a set of data
- find the equation for the line of fit and use it to **make predictions**

When you graph real data, the points sometimes show a linear trend. However, it is very unlikely that all the points will lie exactly on a line. It is up to you to find a line that summarizes, or models, the data. A line that fits a set of data reasonably well is called a **line of fit**.

The guidelines listed under “Finding a Line of Fit” in your book will help you find a line that fits a set of data reasonably well. Once you draw a line of fit, you can write an equation that approximates the relationship between the variables. You can then use the equation to make predictions about values between and beyond the data points.

If you know the slope and $y$-intercept of a line, you can easily write an equation in **intercept form**, $y = a + bx$. When you know only the coordinates of two points on a line or the slope and the coordinates of one point, you can write an equation in **point-slope form**. This form is summarized under “Point-Slope Form” in your book. Read this information carefully.

The example in your book shows how to fit a line to a set of data and then use the line’s equation to make predictions. Work through the example. Note that part b asks you to make a prediction for a value beyond the last year listed in the table. The process of using a model to make a prediction **beyond** the first or last data point is called **extrapolation**. Finding a value **between** given data points is called **interpolation**. So, for example, if you were to predict the CO$_2$ concentration in 1991, you would be using interpolation.

**Investigation: The Wave**

You have probably seen fans at sporting events create a “wave” by standing up quickly in succession with their arms upraised and then sitting down again. In this investigation, you find an equation to model the relationship between the number of people and the length of time it takes to complete the wave.

This table shows the wave data one class collected using the instructions in Step 1.

<table>
<thead>
<tr>
<th>Number of people</th>
<th>2</th>
<th>5</th>
<th>6</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>15</th>
<th>16</th>
<th>22</th>
</tr>
</thead>
<tbody>
<tr>
<td>Time (s)</td>
<td>2.1</td>
<td>4.4</td>
<td>5.2</td>
<td>5.8</td>
<td>4.7</td>
<td>6.7</td>
<td>7.5</td>
<td>10.4</td>
<td>11.0</td>
</tr>
</tbody>
</table>

Use these sample data and your book to work through Steps 2 and 3 of the investigation. Then look at the sample responses on the next page.

(continued)
Lesson 3.3 • Fitting a Line to Data (continued)

Step 2 At right is the graph of the data with a reasonable line of fit. Your line of fit is probably different.

The line passes through (5, 4) and (18, 10), so its slope is \( \frac{10 - 4}{18 - 5} = \frac{6}{13} \).

The point-slope form of the equation (using the point (5, 4)) is

\[ y = 4 + \frac{6}{13}(x - 5) \]

The variable \( \hat{y} \) ("y hat") is used in place of \( y \) to indicate that this is a prediction line.

The slope of the line, \( \frac{6}{13} \), or about 0.46, means that for each new person participating, the amount of time it takes to complete a wave increases by 0.46 second.

To find the \( y \)-intercept, rewrite the equation in intercept form.

\[ \hat{y} = 4 + \frac{6}{13}(x - 5) \]
\[ \hat{y} = \frac{52}{13} + \frac{6}{13}x - \frac{30}{13} \]
\[ \hat{y} = \frac{22}{13} + \frac{6}{13}x \]

The \( y \)-intercept is \( \frac{22}{13} \), or about 1.69. This means that it would take 0 people 1.69 seconds to complete a wave. This does not make sense, so the \( y \)-intercept has no meaning in this context.

To find the \( x \)-intercept, substitute 0 for \( y \) and solve for \( x \).

\[ 0 = \frac{22}{13} + \frac{6}{13}x \]
\[ -\frac{11}{3} = x \]

This means that in 0 second, \( -\frac{11}{3} \) people could complete a wave. This does not make sense, so the \( x \)-intercept has no meaning in this context.

A reasonable domain for the sample data would be from 0 people to 22 people.

Step 3 By this equation, if there were 750 students in a school, then it would take them \( \frac{6}{13}(750) + \frac{22}{13} = 348 \) seconds to complete a wave. It would take 40,000 people in a large stadium \( \frac{6}{13}(40,000) + \frac{22}{13} = 18,463 \) seconds to complete the wave. This is more than 5 hours!

Actually, with a large group of people, the wave gains momentum and begins to travel faster. So, for a large number of people, the data may not be linear.
In this lesson you will

- fit the **median-median line** to a set of data

So far, you have fit lines to data by “eyeballing”—that is, by looking at the pattern of points and drawing a line you think is a good fit. Probably you and your classmates often found different equations for the same set of data.

There are several more formal methods for finding a line of fit. In this lesson, you will learn a procedure for finding the **median-median line**. If you and your classmates follow the procedure correctly, you will all get the same line of fit for a given set of data.

The text before the example in your book explains how to find the median-median line. Read this text and then carefully work through the example. (Do not just read the example; follow along with a pencil and paper.) Then review the steps for finding a median-median line given after the example.

**Investigation: Airline Schedules**

In this investigation you will find a median-median line that models the relationship between the distance of an airline flight and the flight time. Complete the steps of the investigation yourself before reading the text below.

**Step 1** A flight from Detroit to Cincinnati is 64 minutes long and 229 miles.

**Step 2** Below is a graph of the data.

![Graph of flight time vs. distance](image)

**Step 3** Here are the steps for finding the median-median line:

1. Order the data by domain value. Then, divide the data into three groups of equal size. Because the 10 values do not divide evenly into three groups, divide them into groups of 3-4-3.

   Find the median $x$-value and the median $y$-value in each group. The $x$-values are in order, but notice that the Denver flight distance is longer than the Houston flight distance, even though the travel time is shorter. Name the points with these median $x$- and $y$-values $M_1$, $M_2$, and $M_3$, respectively. $M_1$ is (67, 306), $M_2$ is (168, 1015), and $M_3$ is (288, 1979).
Lesson 3.4 • The Median-Median Line (continued)

2. Find the slope of the line through $M_1$ and $M_3$. This will be the slope of the median-median line.

\[
slope = \frac{1979 - 306}{288 - 67} = 7.57
\]

3. Find the equation of the line through $M_1$ with the slope found in Step 2. The equation of the line through $M_3$ will be the same.

\[
y - 306 = 7.57(x - 67) \quad \text{Point-slope form.}
\]

\[
y - 306 = 7.57x - 507.19 \quad \text{Distribute the 7.57.}
\]

\[
\hat{y} = -201.19 + 7.57x \quad \text{Intercept form.}
\]

4. Find the equation of the line through $M_2$ with the slope found in Step 2.

\[
y - 1015 = 7.57(x - 168) \quad \text{Point-slope form.}
\]

\[
y - 1015 = 7.57x - 1271.76 \quad \text{Distribute the 7.57.}
\]

\[
\hat{y} = -256.76 + 7.57x \quad \text{Intercept form.}
\]

5. Find the mean of the $y$-intercepts of the lines through $M_1$, $M_2$, and $M_3$.

(The $y$-intercepts of the lines through $M_1$ and $M_3$ are the same.)

\[
\frac{-201.19 + -201.19 + -256.76}{3} \approx -219.7
\]

So, the equation of the median-median line is \( \hat{y} = -219.7 + 7.57x \).

Step 4  The graph shows points $M_1$, $M_2$, and $M_3$, and the median-median line.

Step 5  Here are sample answers to the questions in a–f:

a. Sample answers: If $x = 300$, then $\hat{y} = -219.7 + 7.57(300) \approx 2051$; in 300 minutes you can fly about 2050 miles from Detroit. If $x = 50$, then $\hat{y} = -219.7 + 7.57(50) = 158.8$; in 50 minutes you can fly about 160 miles from Detroit.

b. (189, 1092) and (64, 229) are the farthest from the median-median line. The flight times may have been influenced by other factors besides distance, such as prevailing winds, the size and type of plane, or the geography.

c. The slope indicates that the distance traveled in 1 minute is approximately 7.57 miles, representing an overall speed of about 7.57 mi/min.

d. The $y$-intercept represents the number of miles traveled at 0 minutes, approximately $-219.7$ miles. This represents the distance a plane would have traveled if it hadn’t been sitting on the runway.

e. The domain is $64 \leq x \leq 303$ min, which includes all the flight times. The range is $229 \leq y \leq 2079$ mi, which includes all the flight distances.

f. It’s easier to fit a line by eyeballing, but the median-median line gives a standard, consistent model of the data.
In this lesson you will

- calculate residuals and the root mean square error and use them to evaluate how well a line fits a set of data.

One way to evaluate how accurately a linear model describes a set of data is to look at the residuals, or the vertical distances between the points in the data set and the points generated by the line of fit.

\[
\text{residual} = y\text{-value of data point} - y\text{-value of point on line}
\]

The closer a point is to the line, the closer its residual will be to zero. A positive residual indicates that the point is above the line. A negative residual indicates that it is below the line. If a line is a good fit, then there will be about as many points above the line as below it, so the sum of the residuals will be near zero.

Study Example A in your book, which shows you how to find and interpret residuals.

**Investigation: Spring Experiment**

**Step 1** The investigation in your book describes an experiment in which you attach various masses to the end of a spring and then measure the length of the spring. These data were collected from such an experiment. Try to complete the steps yourself before reading the text below.

<table>
<thead>
<tr>
<th>Mass (g)</th>
<th>0</th>
<th>50</th>
<th>60</th>
<th>70</th>
<th>80</th>
<th>90</th>
<th>100</th>
<th>110</th>
<th>120</th>
<th>130</th>
<th>140</th>
</tr>
</thead>
<tbody>
<tr>
<td>Spring length (cm)</td>
<td>6.4</td>
<td>8</td>
<td>8.5</td>
<td>9</td>
<td>9.3</td>
<td>9.8</td>
<td>10.1</td>
<td>10.5</td>
<td>10.7</td>
<td>11</td>
<td>11.6</td>
</tr>
</tbody>
</table>

**Step 2** At right is a plot of the data with mass on the x-axis and spring length on the y-axis. The median-median line for the data, \( \hat{y} = 0.037x + 6.29 \), is also graphed.

**Step 3** The slope indicates that the length of the spring increases about 0.037 cm for each additional gram of weight. The y-intercept means that the median-median line predicts that the spring is about 6.29 cm long when no weight is attached.

**Step 4** This table shows the residuals, rounded to the hundredths. (The \( \hat{y} \)-values are calculated using unrounded values for the slope and intercept.)

<table>
<thead>
<tr>
<th>x</th>
<th>0</th>
<th>50</th>
<th>60</th>
<th>70</th>
<th>80</th>
<th>90</th>
<th>100</th>
<th>110</th>
<th>120</th>
<th>130</th>
<th>140</th>
</tr>
</thead>
<tbody>
<tr>
<td>y</td>
<td>6.4</td>
<td>8</td>
<td>8.5</td>
<td>9</td>
<td>9.3</td>
<td>9.8</td>
<td>10.1</td>
<td>10.5</td>
<td>10.7</td>
<td>11</td>
<td>11.6</td>
</tr>
<tr>
<td>( \hat{y} )</td>
<td>6.29</td>
<td>8.15</td>
<td>8.52</td>
<td>8.89</td>
<td>9.26</td>
<td>9.63</td>
<td>10.00</td>
<td>10.38</td>
<td>10.75</td>
<td>11.12</td>
<td>11.49</td>
</tr>
<tr>
<td>( y - \hat{y} )</td>
<td>0.11</td>
<td>-0.15</td>
<td>-0.01</td>
<td>0.11</td>
<td>0.04</td>
<td>0.17</td>
<td>0.10</td>
<td>0.12</td>
<td>-0.05</td>
<td>-0.12</td>
<td>0.11</td>
</tr>
</tbody>
</table>

(continued)
Lesson 3.5 • Prediction and Accuracy (continued)

a. The sum of the residuals is about 0.42, which is fairly small relative to the $y$-values, so the model is a good fit.

b. The greatest positive residual is about 0.17, for 90 g. The negative residual with the greatest magnitude is about $-0.15$, for 50 g. It’s possible that the measurement of the spring length was less accurate for these data values. However, these residuals are not much larger than the other residuals, so it’s unlikely that a large measurement error happened.

c. This model appears to be a good fit for weights up to 140 g. So, if a measured length was more than 0.2 cm off from the prediction, you might want to repeat the measurement. However, the model may not be accurate for much larger weights.

d. The predicted length for a mass of 160 g is about 12.2 cm. Using the largest residual (rounded to 0.2 cm) for the possible error, $12.2 \pm 0.2$ cm is a reasonable prediction for the length.

The graph on page 154 of your book illustrates that it is possible for the sum of the residuals to be close to 0 even if the line is a poor fit. For a line to be a good fit, the individual residuals should also be close to 0. There is, however, a single measure that gives an indication of how well a line fits a data set. This measure is called the **root mean square error**. You calculate the root mean square error by following these steps:

1. Calculate the residuals.
2. Square the residuals.
3. Find the sum of the squares of the residuals.
4. Divide the sum by 2 less than the number of data points.
5. Take the square root of the quotient from the previous step.

To learn more about the root mean square error, read the rest of Lesson 3.5 in your book. Try to solve the problem in Example B *without* looking at the solution, and then check your answer.
In this lesson you will

- write systems of equations to represent real-life situations
- solve systems of equations by using graphs and tables
- solve systems of equations using a simple form of substitution

A set of two or more equations that have the same variables and that are solved or studied simultaneously is called a system of equations. Example A in your book is about a real-world problem that you can solve by finding the solution to a system of equations. The example shows that you can estimate the solution to a system by graphing the equations and finding the point where the graphs intersect or by making a table and looking for the $x$-value for which the $y$-values are the same. Work through Example A carefully.

**Investigation: Population Trends**

Read the investigation in your book and try to solve the problem given in Step 1. When you are finished, read the text below, which describes two possible methods for solving the problem.

You can model the population of San Jose by the median-median line

$$y = -30,850,000 + 15,870x,$$

where $x$ is the year and $y$ is the population.

You can model the population of Detroit by the median-median line

$$y = 36,331,000 - 17,700x,$$

where $x$ is the year and $y$ is the population.

You can use a graph or a table to estimate the year the two cities had the same population. The graph shows that the population of San Jose equaled that of Detroit between 2001 and 2002. In both cities, the population was about 909,000.

The table shows that in 2001, the population of Detroit was greater than that of San Jose, but in 2002, the population of Detroit was less. That means that the two cities must have had the same population between 2001 and 2002. To get an accurate estimate, you would need to show smaller increments in the table.
Both methods found the same answer. The solution method you choose will depend on how accurate your answer needs to be and whether you have technology available.

You have seen that you can estimate a solution to a system by using a graph or a table. In many cases you can find an exact solution by using symbolic methods.

Example B in your book demonstrates one method. You will learn other methods in the next lesson. Work through Example B carefully, and read the text after Example B. Then read the example below.

**EXAMPLE**

Josie makes and sells silver earrings. She rented a booth at a weekend art fair for $325. The materials for each pair of earrings cost $6.75, and she sells each pair for $23. How many pairs does she need to sell at the fair in order to break even?

**Solution**

If \( x \) is the number of earrings, then you can write these equations:

\[
\begin{align*}
y &= 325 + 6.75x \quad \text{Josie’s expenses.} \\
y &= 23x \quad \text{Josie’s income.}
\end{align*}
\]

The graph shows that Josie’s income eventually exceeds her expenses.

The intersection represents the break-even point, when Josie’s income equals her expenses. You can find the break-even point by tracing the graph or using a table. You can also solve a system of equations. Set the right sides of the equations equal to each other and solve for \( x \).

\[
325 + 6.75x = 23x \quad \text{When Josie’s expenses equal her income.}
\]

\[
325 = 16.25x \quad \text{Subtract 6.75x from both sides.}
\]

\[
20 = x \quad \text{Divide both sides by 16.25.}
\]

Josie needs to sell 20 pairs of earrings in order to break even.
In this lesson you will

- use the substitution method to solve systems of equations
- use the elimination method to solve systems of equations

This lesson discusses two methods of solving systems of equations, the substitution method and the elimination method. You probably learned these techniques in a previous math class. To review and practice these methods, read the text up to the investigation in your book. Then, read the example below.

**EXAMPLE**

Solve this system for \( x \) and \( y \).

\[
\begin{align*}
  y - 5 &= -3x \\
  7x + 3y &= 7
\end{align*}
\]

**Solution**

Solve the first equation for \( y \): \( y = 5 - 3x \). Now, substitute \( 5 - 3x \) for \( y \) in the second equation.

\[
\begin{align*}
  7x + 3y &= 7 & \text{The second equation.} \\
  7x + 3(5 - 3x) &= 7 & \text{Substitute } 5 - 3x \text{ for } y. \\
  7x + 15 - 9x &= 7 & \text{Distribute 3.} \\
  -2x &= -8 & \text{Subtract 15 from both sides and combine like terms.} \\
  x &= 4 & \text{Divide both sides by } -2.
\end{align*}
\]

Now that you know the value of \( x \), substitute it into either equation to find the value of \( y \).

\[
\begin{align*}
  y - 5 &= -3(4) & \text{Substitute 4 for } x \text{ in the first equation.} \\
  y &= -7 & \text{Multiply and add 5 to both sides.}
\end{align*}
\]

The solution to the system is \((4, -7)\).

**Investigation: What’s Your System?**

In this investigation, you will explore the properties of different kinds of systems. Work through the steps of the investigation on your own. Then read the solutions below.

**Step 1** Use the elimination method to solve each system.

a. Subtract the equations to eliminate \( x \). The resulting equation is \( 8y = -16 \)
   so \( y = -2 \). Substitute \( -2 \) for \( y \) in the first equation to get \( 2x + 5(-2) = 6 \),
   and solve for \( x \). \( 2x = 16 \), so \( x = 8 \). The solution is \((8, -2)\).

b. Multiply the first equation by 2 to get \( 6x + 4y = 24 \). Add the resulting equation to the second equation. This gives the equation \( 0 = 0 \).
   This equation is true for any values of \( x \) and \( y \), so there are infinitely many solutions.

(continued)
Lesson 3.7 • Substitution and Elimination (continued)

c. To eliminate \( x \), multiply both equations so that \( x \) has the coefficient 12 (or \(-12\)), which is a common multiple of 4 and 3.

\[
3(4x - 8y = 5) \rightarrow 12x - 24y = 15 \quad \text{Multiply the first equation by 3.}
\]
\[
4(-3x + 6y = 11) \rightarrow -12x + 24y = 44 \quad \text{Multiply the second equation by 4.}
\]

\[
0 = 59 \quad \text{Add the resulting equations.}
\]

The final equation, \( 0 = 59 \), is false for all values of \( x \) and \( y \), so there are no solutions to the system.

d. Multiply the first equation by 3 to get \(-6x + 3y = 15\). Add the resulting equation to the second equation to get \( 0 = 0 \). As in 1b, there are infinitely many solutions.

e. Add the two equations to eliminate \( y \). The resulting equation is \( 6x = 12 \), so \( x = 2 \). Substitute 2 for \( x \) in the first equation and solve for \( y \): \( 2 + 3y = 6 \), so \( y = \frac{4}{3} \). The solution is \( (2, \frac{4}{3}) \).

f. Multiply the first equation by \(-3\) to get \(-3x - 9y = -24\). Add the resulting equation to the second equation to eliminate \( x \). The resulting equation, \( 0 = -28 \), is false, so the system has no solution.
Lesson 3.7 • Substitution and Elimination (continued)

2e. 

Step 3  The systems from parts a, b, d, and e are consistent.

Step 4  The systems from parts c and f are inconsistent.

Step 5  The systems from parts b and d are dependent and the systems from parts a and e are independent.

Step 6  If the elimination method results in an equation that is always true, like \(0 = 0\), then the system is consistent and dependent (the lines are the same). If the elimination method results in an answer for \(x\) and \(y\), then the system is consistent and independent (there is a single intersection point). If the final equation is untrue, such as \(0 = 24\), then the system is inconsistent (the lines are parallel).