**Content Summary**

In Chapter 8, students continue their work with functions, especially nonlinear functions, through further study of function graphs. In particular, they consider three ways of changing the location, orientation, and size of those graphs. (Note: You might skip the material on matrices if your student’s teacher is not covering Lesson 8.7; matrices are often covered as part of an advanced algebra curriculum.)

**Translations**

A translation shifts points or graphs on the plane. If a point \((x, y)\) is shifted to the right \(h\) units and up \(k\) units, the resulting point is \((x + h, y + k)\).

A function graph can be shifted in the same way, by replacing \(x\) with \(x - h\) in each occurrence of \(x\) in the graph’s equation, and by replacing \(y\) with \(y - k\) in each occurrence of \(y\).

As an example, consider the rational function \(f(x) = \frac{1}{x}\), which the book introduces in this chapter. If you shift the graph of \(y = \frac{1}{x}\) to the left 2 units and up 3 units, the result has the equation \(y - 3 = \frac{1}{x + 2}\). You can also think of the resulting equation as the graph of function \(g(x) = \frac{1}{x + 2} + 3\). Although \(f(x) = \frac{1}{x}\) is undefined for \(x = 0\) and has \(y = 0\) as an asymptote (a line that the graph approaches but never touches), \(g(x)\) is undefined 2 units to the left, where \(x = -2\), and has an asymptote 3 units higher, at \(y = 3\).

![Graph showing translation of a rational function](image)

It is possible to represent transformations with matrices. In particular, a translation can be represented by matrix addition. To do so, you can represent the point \((x, y)\) by the \(2 \times 1\) matrix \[
\begin{bmatrix}
x \\
y
\end{bmatrix}
\]
and the translation by \[
\begin{bmatrix}
h \\
k
\end{bmatrix}
\]. Then the image point is represented by the sum of these matrices:
\[
\begin{bmatrix}
x \\
y
\end{bmatrix}
+ \begin{bmatrix}
h \\
k
\end{bmatrix}
= \begin{bmatrix}
x + h \\
y + k
\end{bmatrix}.
\]

In fact, this matrix approach can represent the translation of more than one point. *Discovering Algebra* shows how a matrix can represent the vertices (corners) of a polygon, with each column being the coordinates of a vertex. For example, the matrix
\[
\begin{bmatrix}
3 & 1 & -2 & -1 & 2 \\
2 & -1 & 0 & 2 & 4
\end{bmatrix}
\]
represents the pentagon pictured on the next page.

A shift to the right 3 units and up 2 units can be represented by the matrix
\[
\begin{bmatrix}
3 & 3 & 3 & 3 & 3 \\
2 & 2 & 2 & 2 & 2
\end{bmatrix}.
\]

(continued)
The resulting polygon is represented by the sum of the matrices:
\[
\begin{bmatrix}
3 & 1 & -2 & -1 & 2 \\
2 & -1 & 0 & 2 & 4
\end{bmatrix}
+ \begin{bmatrix}
3 & 3 & 3 & 3 & 3 \\
2 & 2 & 2 & 2 & 2
\end{bmatrix}
= \begin{bmatrix}
6 & 4 & 1 & 2 & 5 \\
4 & 1 & 2 & 4 & 6
\end{bmatrix}
\]
It is shown as the dashed polygon on the graph.

**Reflections**
The book examines *reflections* (or flips) of points and graphs across the axes.

When you reflect a point \((x, y)\) across the \(y\)-axis, the result is the point \((-x, y)\).

Reflecting the point \((x, y)\) across the \(x\)-axis yields an image of \((x, -y)\).

To reflect a graph across the \(y\)-axis, you replace each occurrence of \(x\) in its equation with \(-x\). To reflect a graph across the \(x\)-axis, you replace \(y\) with \(-y\). For example, the reflection of the graph of \(y = \frac{1}{x + 2}\) across the \(y\)-axis has equation \(f(x) = \frac{1}{-x + 2}\).

The reflection of \(y = \frac{1}{x + 2}\) across the \(x\)-axis has equation \(-y = \frac{1}{x + 2}\), or \(y = -\frac{1}{x + 2}\).

Reflections can also be represented by matrix multiplication. For example, to reflect the pentagon represented by
\[
\begin{bmatrix}
3 & 1 & -2 & -1 & 2 \\
2 & -1 & 0 & 2 & 4
\end{bmatrix}
\]
across the \(y\)-axis, multiply by the matrix
\[
\begin{bmatrix}
-1 & 0 \\
0 & 1
\end{bmatrix}
\]
to get
\[
\begin{bmatrix}
-1 & 0 \\
0 & 1
\end{bmatrix}
\begin{bmatrix}
3 & 1 & -2 & -1 & 2 \\
2 & -1 & 0 & 2 & 4
\end{bmatrix}
= \begin{bmatrix}
-3 & -1 & 2 & 1 & -2 \\
2 & -1 & 0 & 2 & 4
\end{bmatrix}
\]

**Stretches and Shrinks**
A vertical stretch by positive factor \(a\) changes \((x, y)\) to \((x, ay)\). If \(a\) is less than 1 (but still positive), the stretch is a shrink.

To stretch or shrink a graph vertically by positive factor \(a\), replace each occurrence of \(y\) with \(\frac{y}{a}\) in the equation of the graph. For example, replacing \(y\) with \(\frac{y}{2}\) in the equation \(f(x) = \frac{1}{x}\) creates a vertical stretch by a factor of 2. This is the equation of the function \(g(x) = \frac{2}{x}\). (continued)
Chapter 8 • Transformations (continued)

Stretches and shrinks can also be represented by matrix multiplication.

You can find the image of pentagon \[
\begin{bmatrix}
3 & 1 & -2 & -1 & 2 \\
2 & -1 & 0 & 2 & 4
\end{bmatrix}
\] after it is stretched vertically by a factor of \(\frac{4}{3}\) if you multiply by the matrix \[
\begin{bmatrix}
1 & 0 \\
0 & \frac{4}{3}
\end{bmatrix}
\] to get \[
\begin{bmatrix}
1 & 0 \\
0 & \frac{4}{3}
\end{bmatrix}
\begin{bmatrix}
3 & 1 & -2 & -1 & 2 \\
2 & -1 & 0 & 2 & 4
\end{bmatrix}
= \begin{bmatrix}
\frac{2}{3} & -\frac{1}{3} & 0 & \frac{2}{3} & \frac{4}{3}
\end{bmatrix}.
\]

Summary Problem

Use transformations of the function \(f(x) = x\) to fit the data in the preceding table as well as you can.

Questions you might ask in your role as student to your student include:

- Why do different translations give the same result?
- Why do different reflections give the same result?
- Why do different combinations of stretches and shrinks give the same result?
- Does the order in which you do the transformations matter?
- Could a starting function other than \(f(x) = x\) be transformed to fit the data points better?

Sample Answers

For a line with equation \(y = a + bx\), replacing \(x\) with \(x - h\) gives \(y = a + bx - bh\), or \(y + bh = a + bx\). So a horizontal translation by \(h\) is the same as a vertical translation by \(-bh\). Algebra also shows that a reflection of a line across the \(x\)-axis can be accomplished by a reflection across the \(y\)-axis, combined with a translation, and that stretching a line in one direction is equivalent to shrinking it in the perpendicular direction. A nonlinear function will probably fit the data better.
Chapter 8 • Review Exercises

1. (Lessons 8.1, 8.3, 8.4) Draw this triangle on graph paper or on your calculator. Then draw the image under each of the following transformations. Describe each transformation.
   a. \((x - 2, y + 1)\)  
   b. \((-x, y)\)  
   c. \((x, -y)\)  
   d. \((0.5x, 3y)\)

2. (Lessons 8.2–8.4) The graph of the function \(y = |x - 3|\) is shown below. Name the functions that give the following transformations of the graph. Check each answer by graphing it on your calculator.
   a. Translate right 2 units.
   b. Reflect across the \(y\)-axis and translate up 1 unit.
   c. Reflect across the \(x\)-axis, shrink vertically by a factor of 0.5, translate left 1 unit, and translate up 3 units.

3. (Lesson 8.6) Reduce each expression to lowest terms. State any restrictions on the variable.
   a. \(\frac{(2x^2)(10x^4)}{4x^3}\)  
   b. \(\frac{2x^2 - 4x}{6x}\)  
   c. \(\frac{4x(x - 3)}{2(x - 3)^2}\)  
   d. \(\frac{6 - 2x^2}{2x}\)
1. a. Translate left 2 units and up 1 unit.
   b. Reflect across the y-axis.
   c. Reflect across the x-axis.
   d. Shrink horizontally by a factor of 0.5; stretch vertically by a factor of 3.

![Graph Image]

2. a. \( y = |x - 3| \) Original function.
   \[
   y = |(x - 2) - 3| \quad \text{Replace } x \text{ with } x - 2 \text{ to translate the graph right 2 units.}
   \]
   \[
   y = |x - 5| \quad \text{Translate left 2 units and up 1 unit.}
   \]

![Graph Image]

b. \( y = |x - 3| \) Original function.
   \[
   y = |(-x) - 3| \quad \text{Replace } x \text{ with } -x \text{ to reflect the graph across the y-axis.}
   \]
   \[
   y - 1 = |(-x) - 3| \quad \text{Replace } y \text{ with } y - 1 \text{ to translate the graph up 1 unit.}
   \]

![Graph Image]

c. \( y = |x - 3| \) Original function.
   \[
   -y = |x - 3| \quad \text{Replace } y \text{ with } -y \text{ to reflect across the x-axis.}
   \]
   \[
   y = -|x - 3| \quad \text{Solve for } y.
   \]
   \[
   2y = -|x - 3| \quad \text{Replace } y \text{ with } \frac{y}{0.5}, \text{ or } 2y, \text{ to shrink the graph vertically by a factor of 0.5.}
   \]

![Graph Image]

3. a. \[
\frac{(2x^2)(10x^4)}{4x^3} = \frac{2 \cdot 2 \cdot 5 \cdot x^6}{2 \cdot 2 \cdot x^3} = 5x^3, \text{ where } x \neq 0
\]
The restriction \( x \neq 0 \) is necessary because the \( x \)-value 0 would make the denominator of the original expression zero.

b. \[
2x^2 - 4x = \frac{2x(x - 2)}{2 \cdot 3 \cdot x} = \frac{x - 2}{3}, \text{ where } x \neq 0
\]
The restriction \( x \neq 0 \) is necessary because the \( x \)-value 0 would make the denominator of the original expression zero.

c. \[
\frac{4x(x - 3)}{2(x - 3)^2} = \frac{2x}{x - 3}, \text{ where } x \neq 3.
\]

d. \[
\frac{6 - 2x^2}{2x} = \frac{2(3 - x^2)}{2 \cdot x} = \frac{3 - x^2}{x}, \text{ where } x \neq 0.
\]