Applications of Trigonometric and Circular Functions

Stresses in the earth compress rock formations and cause them to buckle into sinusoidal shapes. It is important for geologists to be able to predict the depth of a rock formation at a given point. Such information can be very useful for structural engineers as well. In this chapter you’ll learn about the circular functions, which are closely related to the trigonometric functions. Geologists and engineers use these functions as mathematical models to perform calculations for such wavy rock formations.
Mathematical Overview

So far you’ve learned about transformations and sinusoids. In this chapter you’ll combine what you’ve learned so that you can write a particular equation for a sinusoid that fits any given conditions. You will approach this in four ways.

**Graphically** The graph is a sinusoid that is a cosine function transformed through vertical and horizontal translations and dilations. The independent variable here is \( x \) rather than \( \theta \) so that you can fit sinusoids to situations that do not involve angles.

**Algebraically** Particular equation: \( y = 7 + 2 \cos \frac{\pi}{3}(x - 1) \)

**Numerically**

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**Verbally** The circular functions are just like the trigonometric functions except that the independent variable is an arc of a unit circle instead of an angle. Angles in radians form the link between angles in degrees and numbers of units of arc length.
3-1 Sinusoids: Amplitude, Period, and Cycles

Figure 3-1a shows a dilated and translated sinusoid and some of its graphical features. In this section you will learn how these features relate to transformations you’ve already learned.

Figure 3-1a

**OBJECTIVE**

Learn the meanings of amplitude, period, phase displacement, and cycle of a sinusoidal graph.

### Exploratory Problem Set 3-1

1. Sketch one cycle of the graph of the parent sinusoid $y = \cos \theta$, starting at $\theta = 0^\circ$. What is the amplitude of this graph?

2. Plot the graph of the transformed cosine function $y = 5 \cos \theta$. What is the amplitude of this graph? What is the relationship between the amplitude and the vertical dilation of a sinusoid?

3. What is the period of the transformed function in Problem 2? What is the period of the parent function $y = \cos \theta$?

4. Plot the graph of $y = \cos 3\theta$. What is the period of this transformed function graph? How is the 3 related to the transformation? How could you calculate the period using the 3?

5. Plot the graph of $y = \cos (\theta - 60^\circ)$. What transformation is caused by the $60^\circ$?

6. The $(\theta - 60^\circ)$ in Problem 5 is called the argument of the cosine. The phase displacement is the value of $\theta$ that makes the argument equal zero. What is the phase displacement for this function? How is the phase displacement related to the horizontal translation?
3-2 General Sinusoidal Graphs

In Section 3-1, you encountered the terms period, amplitude, cycle, phase displacement, and sinusoidal axis. They are often used to describe horizontal and vertical translation and dilation of sinusoids. In this section you’ll make the connection between the new terms and these transformations so that you will be able to fit an equation to any given sinusoid. This in turn will help you use sinusoidal functions as mathematical models for real-world applications such as the variation of average daily temperature with the time of year.

Recall from Chapter 2 that the period of a sinusoid is the number of degrees per cycle. The reciprocal of the period, or the number of cycles per degree, is called the frequency. It is convenient to use the frequency when the period is very short. For instance, the alternating electrical current in the United States has a frequency of 60 cycles per second, meaning that the period is 1/60 second per cycle.

You can see how the general sinusoidal equations allow for all four transformations.

**DEFINITION: General Sinusoidal Equation**

\[ y = C + A \cos B(\theta - D) \quad \text{or} \quad y = C + A \sin B(\theta - D), \]

where

- \(|A|\) is the amplitude (\(A\) is the vertical dilation, which can be positive or negative).
- \(B\) is the reciprocal of the horizontal dilation.
- \(C\) is the location of the sinusoidal axis (vertical translation).
- \(D\) is the phase displacement (horizontal translation).

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7. Plot the graph of \(y = 6 + \cos \theta\). What transformation is caused by the 6?

8. The sinusoidal axis runs along the middle of the graph of a sinusoid. It is the dashed centerline in Figure 3-1a. What transformation of the function \(y = \cos x\) does the location of the sinusoidal axis indicate?

9. What are the amplitude, period, phase displacement, and sinusoidal axis location of the graph of \(y = 6 + 5 \cos 3(\theta - 60^\circ)\)? Check by plotting on your grapher.
The period can be calculated from the value of $B$. Because $\frac{1}{B}$ is the horizontal dilation and because the parent cosine and sine functions have the period $360^\circ$, the period of a sinusoid equals $\frac{1}{|B|}(360^\circ)$. Dilations can be positive or negative, so you must use the absolute value symbol.

**PROPERTY: Period and Frequency of a Sinusoid**

For general equations $y = C + A \cos B(\theta - D)$ or $y = C + A \sin B(\theta - D)$

\[
\text{period} = \frac{1}{|B|}(360^\circ) \quad \text{and} \quad \text{frequency} = \frac{1}{\text{period}} \cdot \frac{|B|}{360^\circ}
\]

Next you'll use these properties and the general equation to graph sinusoids and find their equations.

**Background: Concavity, Points of Inflection, and Upper and Lower Bounds**

A smoothly curved graph can have a **concave** (hollowed-out) side and a **convex** (bulging) side, as Figure 3-2a shows for a typical sinusoid. In calculus, for reasons you will learn, mathematicians usually refer to the concave side. Figure 3-2a also shows regions where the concave side of the graph is up or down. A **point of inflection** occurs where a graph stops being concave one way and starts being concave the other way. The word originates from the British spelling, *inflexion*, which means “not flexed.”

As you can see from Figure 3-2b, the sinusoidal axis goes through the points of inflection. The lines through the high points and the low points are called the **upper bound** and the **lower bound**, respectively. The high points and low points are called **critical points** because they have a “critical” influence on the size and location of the sinusoid. Note that it is a quarter-cycle between a critical point and the next point of inflection.
EXAMPLE 1

Suppose that a sinusoid has period 12° per cycle, amplitude 7 units, phase displacement −4° with respect to the parent cosine function, and a sinusoidal axis 5 units below the θ-axis. Without using your grapher, sketch this sinusoid and then find an equation for it. Verify with your grapher that your equation and the sinusoid you sketched agree with each other.

Solution

First draw the sinusoidal axis at \( y = -5 \), as in Figure 3-2c. (The long-and-short dashed line is used by draftspersons for centerlines.) Use the amplitude, 7, to draw the upper and lower bounds 7 units above and 7 units below the sinusoidal axis.

Next find some critical points on the graph (Figure 3-2d). Start at \( \theta = -4° \), because that is the phase displacement, and mark a high point on the upper bound. (The cosine function starts a cycle at a high point because \( \cos 0° = 1 \).) Then use the period, 12°, to plot the ends of the next two cycles.

\[-4° + 12° = 8°\]
\[-4° + 2(12°) = 20°\]

Mark some low critical points halfway between consecutive high points.

Now mark the points of inflection (Figure 3-2e). They lie on the sinusoidal axis, halfway between consecutive high and low points.
Finally, sketch the graph in Figure 3-2f by connecting the critical points and points of inflection with a smooth curve. Be sure that the graph is rounded at the critical points and that it changes concavity at the points of inflection.

Because the period of this sinusoid is 12° and the period of the parent cosine function is 360°, the horizontal dilation is

$$\text{dilation} = \frac{12\degree}{360\degree} = \frac{1}{30}$$

The coefficient $B$ in the sinusoidal equation is the reciprocal of $\frac{1}{30}$, namely, 30. The horizontal translation is $-4\degree$. Thus a particular equation is

$$y = -5 + 7 \cos (30(\theta + 4))$$

Plotting the graph on your grapher confirms that this equation produces the correct graph (Figure 3-2g).

**EXAMPLE 2**

For the sinusoid in Figure 3-2h, give the period, frequency, amplitude, phase displacement, and sinusoidal axis location. Write a particular equation of the sinusoid. Check your equation by plotting it on your grapher.
Solution

As you will see later, you can use either the sine or the cosine as the pre-image function. Here, use the cosine function, because its “first” cycle starts at a high point and two high points are known.

- To find the period, look at the cycle shown in Figure 3-2h. It starts at 3° and ends at 23°, so the period is 23° − 3°, or 20°.
- The frequency is the reciprocal of the period, 1/20 cycle per degree.
- The sinusoidal axis is halfway between the upper and lower bounds, so y = 1/4(-38 + 56), or 9.
- The amplitude is the distance between the upper or lower bound and the sinusoidal axis.

\[ A = 56 - 9 = 47 \]

- Using the cosine function as the parent function, the phase displacement is 3°. (You could also use 23° or −17°.)
- The horizontal dilation is \( \frac{20}{180} \), so \( B = \frac{360}{180} \) or 18 (the reciprocal of the horizontal dilation). So a particular equation is

\[ y = 9 + 47 \cos 18(\theta - 3°) \]

Plotting the corresponding graph on your grapher confirms that the equation is correct.

You can find an equation of a sinusoid when only part of a cycle is given. The next example shows you how to do this.

EXAMPLE 3

Figure 3-2i shows a quarter-cycle of a sinusoid. Write a particular equation and check it by plotting it on your grapher.

![Figure 3-2i](image)

Solution

Imagine the entire cycle from the part of the graph that is shown. You can tell that a low point is at \( \theta = 24° \) because the graph appears to level out there. So the lower bound is at \( y = 3 \). The point at \( \theta = 17° \) must be an inflection point on the sinusoidal axis at \( y = 8 \) because the graph is a quarter-cycle. So the amplitude is 8 − 3, or 5. Sketch the lower bound, the sinusoidal axis, and the upper bound. Next locate a high point. Each quarter-cycle covers \( 24° - 17° \), or 7°,
so the critical points and points of inflection are spaced $7^\circ$ apart. Thus a high point is at $\theta = 17^\circ - 7^\circ$, or $10^\circ$. Sketch at least one complete cycle of the graph (Figure 3-2j).

The period is $4(7^\circ)$, or $28^\circ$, because a quarter of the period is $7^\circ$. The horizontal dilation is $\frac{2\pi}{3\pi}$, or $\frac{2}{3\pi}$.

The coefficient $B$ in the sinusoidal equation is the reciprocal of this horizontal dilation. If you use the techniques of Example 2, a particular equation is

$$y = 8 + 5 \cos \frac{\pi}{2}(\theta - 10^\circ)$$

Plotting the graph on your grapher shows that the equation is correct.

Note that in all the examples so far a particular equation is used, not the. There are many equivalent forms of the equation, depending on which cycle you pick for the “first” cycle and whether you use the parent sine or cosine function. The next example shows some possibilities.

**Example 4**

For the sinusoid in Figure 3-2k, write a particular equation using

a. Cosine, with a phase displacement other than $10^\circ$

b. Sine

c. Cosine, with a negative vertical dilation factor

d. Sine, with a negative vertical dilation factor

Confirm on your grapher that all four equations give the same graph.

**Solution**

a. Notice that the sinusoid is the same one as in Example 3. To find a different phase displacement, look for another high point. A convenient
one is at $\theta = 38^\circ$. All the other constants remain the same. So another particular equation is

$$y = 8 + 5 \cos \frac{90}{7}(\theta - 38^\circ)$$

b. The graph of the parent sine function starts at a point of inflection on the sinusoidal axis while going up. Two possible starting points appear in Figure 3-2k, one at $\theta = 3^\circ$ and another at $\theta = 31^\circ$.  

$$y = 8 + 5 \sin \frac{90}{7}(\theta - 3^\circ) \quad \text{or} \quad y = 8 + 5 \sin \frac{90}{7}(\theta - 31^\circ)$$

c. Changing the vertical dilation factor from 5 to −5 causes the sinusoid to be reflected across the sinusoidal axis. If you use −5, the “first” cycle starts as a low point instead of a high point. The most convenient low point in this case is at $\theta = 24^\circ$.

$$y = 8 - 5 \cos \frac{90}{7}(\theta - 24^\circ)$$

d. With a negative dilation factor, the sine function starts a cycle at a point of inflection while going down. One such point is shown in Figure 3-2k at $\theta = 17^\circ$.

$$y = 8 - 5 \sin \frac{90}{7}(\theta - 17^\circ)$$

Plotting these four equations on your grapher reveals only one image. The graphs are superimposed on one another.

**Problem Set 3-2**

**Reading Analysis**

From what you have read in this section, what do you consider to be the main idea? How are the words period, frequency, and cycle related to one another in connection with sinusoids? What is the difference between the way $\theta$ appears on the graph of a sinusoid and the way it appears in a uv-coordinate system, as in Chapter 2? How can there be more than one particular equation for a given sinusoid?

**Quick Review**

Problems Q1–Q5 refer to Figure 3-2l.

Q1. How many cycles are there between $\theta = 20^\circ$ and $\theta = 80^\circ$?

Q2. What is the amplitude?

Q3. What is the period?

Q4. What is the vertical translation?

Q5. What is the horizontal translation (for cosine)?

Q6. Find the exact value (no decimals) of $\sin 60^\circ$.

Q7. Find the approximate value of $\sec 71^\circ$.

Q8. Find the approximate value of $\cot^{-1} 4.3$.

Q9. Find the measure of the larger acute angle of a right triangle with legs of lengths 11 ft and 9 ft.

Q10. Expand: $(3x - 5)^2$

For Problems 1–4, find the amplitude, period, phase displacement, and sinusoidal axis location. Without using your grapher, sketch the graph by

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**Figure 3-2l**

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Chapter 3: Applications of Trigonometric and Circular Functions
locating critical points. Then check your graph using your grapher.

1. \( y = 7 + 4 \cos (3(\theta + 10^\circ)) \)
2. \( y = 3 + 5 \cos \frac{1}{2}(\theta - 240^\circ) \)
3. \( y = -10 + 20 \sin \frac{1}{2}(\theta - 120^\circ) \)
4. \( y = -8 + 10 \sin (5(\theta + 6^\circ)) \)

For Problems 5–8,

a. Find a particular equation for the sinusoid using cosine or sine, whichever seems easier.

b. Give the amplitude, period, frequency, phase displacement, and sinusoidal axis location.

c. Use the equation from part a to calculate \( y \) for the given values of \( \theta \). Show that the result agrees with the given graph for the first value.

5. \( \theta = 60^\circ \) and \( \theta = 1234^\circ \)

6. \( \theta = 10^\circ \) and \( \theta = 453^\circ \)

7. \( \theta = 70^\circ \) and \( \theta = 491^\circ \)

8. \( \theta = 8^\circ \) and \( \theta = 1776^\circ \)

For Problems 9–14, find a particular equation of the sinusoid that is graphed.

9.

10.

11.

Section 3-2: General Sinusoidal Graphs
In Problems 15 and 16, a half-cycle of a sinusoid is shown. Find a particular equation of the sinusoid.

15. 

16. 

In Problems 17 and 18, a quarter-cycle of a sinusoid is shown. Find a particular equation of the sinusoid.

17. 

18. 

19. If the sinusoid in Problem 17 is extended to $\theta = 300^\circ$, what is the value of $y$? If the sinusoid is extended to $\theta = 5678^\circ$, is the point on the graph above or below the sinusoidal axis? How far?

20. If the sinusoid in Problem 18 is extended to the left to $\theta = 2.5^\circ$, what is the value of $y$? If the sinusoid is extended to $\theta = 328^\circ$, is the point on the graph above or below the sinusoidal axis? How far?

For Problems 21 and 22, sketch the sinusoid described and write a particular equation of it. Check the equation on your grapher to make sure it produces the graph you sketched.

21. The period equals $72^\circ$, amplitude is 3 units, phase displacement (for $y = \cos \theta$) equals $6^\circ$, and the sinusoidal axis is at $y = 4$ units.

22. The frequency is $\frac{1}{6}$ cycle per degree, amplitude equals 2 units, phase displacement (for $y = \cos \theta$) equals $-3^\circ$, and the sinusoidal axis is at $y = -5$ units.

For Problems 23 and 24, write four different particular equations for the given sinusoid, using

a. Cosine as the parent function with positive vertical dilation
b. Cosine as the parent function with negative vertical dilation
c. Sine as the parent function with positive vertical dilation
d. Sine as the parent function with negative vertical dilation

Plot all four equations on the same screen on your grapher to confirm that the graphs are the same.
23. **Frequency Problem:** The unit for the period of a sinusoid is degrees per cycle. The unit for the frequency is cycles per degree.

a. Suppose that a sinusoid has period \( \frac{\text{degree}}{\text{cycle}} \). What would the frequency be? Why might people prefer to speak of the frequency of such a sinusoid rather than the period?

b. For \( y = \cos 30\theta \), what is the period? What is the frequency? How can you calculate the frequency quickly, using the 300?

24. **Inflection Point Problem:** Sketch the graph of a function that has high and low critical points. On the sketch, show

a. A point of inflection

b. A region where the graph is concave up

c. A region where the graph is concave down

25. **Frequency Problem:** The unit for the period of a sinusoid is degrees per cycle. The unit for the frequency is cycles per degree.

a. Suppose that a sinusoid has period \( \frac{\text{degree}}{\text{cycle}} \). What would the frequency be? Why might people prefer to speak of the frequency of such a sinusoid rather than the period?

b. For \( y = \cos 30\theta \), what is the period? What is the frequency? How can you calculate the frequency quickly, using the 300?

26. **Inflection Point Problem:** Sketch the graph of a function that has high and low critical points. On the sketch, show

a. A point of inflection

b. A region where the graph is concave up

c. A region where the graph is concave down

27. **Horizontal vs. Vertical Transformations Problem:** In the function

\[ y = 3 + 4 \cos 2(\theta - 5) \]

the 3 and the 4 are the vertical transformations, but the 2 and the -5 are the reciprocal and opposite of the horizontal transformations.

a. Show that you can transform the given equation to

\[ \frac{y - 3}{4} = \cos \left( \frac{\theta - 5}{1/2} \right) \]

b. Examine the equation in part a for the transformations that are applied to the \( x \) - and \( y \) -variables. What is the form of these transformations?

c. Why is the original form of the equation more useful than the form in part a?

28. **Journal Problem:** Update your journal with things you have learned about sinusoids. In particular, explain how the amplitude, period, phase displacement, frequency, and sinusoidal axis location are related to the four constants in the general sinusoidal equation. What is meant by critical points, concavity, and points of inflection?

### 3-3 Graphs of Tangent, Cotangent, Secant, and Cosecant Functions

If you enter \( \tan 90^\circ \) into your calculator, you will get an error message because tangent is defined as a quotient. On the unit circle, a point on the terminal side of a \( 90^\circ \) angle has horizontal coordinate zero and vertical coordinate 1. Division of a nonzero number by zero is undefined, which you’ll see leads to **vertical asymptotes** at angle measures for which division by zero would occur. In this section you’ll also see that the graphs of the tangent, cotangent, secant, and cosecant functions are **discontinuous** where the function value would involve division by zero.
You can plot cotangent, secant, and cosecant by using the fact that they are reciprocals of tangent, cosine, and sine, respectively.

\[
\cot \theta = \frac{1}{\tan \theta} \quad \sec \theta = \frac{1}{\cos \theta} \quad \csc \theta = \frac{1}{\sin \theta}
\]

Figure 3-3a shows the graphs of \( y = \tan \theta \) and \( y = \cot \theta \), and Figure 3-3b shows the graphs of \( y = \sec \theta \) and \( y = \csc \theta \), all as they might appear on your grapher. If you use a friendly window that includes multiples of 90° as grid points, you’ll see that the graphs are discontinuous. Notice that the graphs go off to infinity (positive or negative) at odd or even multiples of 90°, exactly those places where the functions are undefined.

To see why the graphs have these shapes, it helps to look at transformations performed on the parent cosine and sine graphs.

**Example 1**

Sketch the graph of the parent sine function, \( y = \sin \theta \). Use the fact that \( \csc \theta = \frac{1}{\sin \theta} \) to sketch the graph of the cosecant function. Show how the asymptotes of the cosecant function are related to the graph of the sine function.

**Solution**

Sketch the sine graph as in Figure 3-3c. Where the value of the sine function is zero, the cosecant function will be undefined because of division by zero. Draw vertical asymptotes at these values of \( \theta \).

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**Objective**

Plot the graphs of the tangent, cotangent, secant, and cosecant functions, showing their behavior when the function value is undefined.
Where the sine function equals 1 or \(-1\), so does the cosecant function, because the reciprocal of 1 is 1 and the reciprocal of \(-1\) is \(-1\). Mark these points as in Figure 3-3d. As the sine gets smaller, the cosecant gets bigger, and vice versa. For instance, the reciprocal of 0.2 is 5. The reciprocal of \(-0.5\) is \(-2\). Sketch the graph consistent with these facts, as in Figure 3-3d.

To understand why the graphs of the tangent and cotangent functions have the shapes in Figure 3-3a, it helps to examine how these functions are related to the sine and cosine functions. By definition,

\[
\tan \theta = \frac{v}{u}
\]

Dividing the numerator and the denominator by \(r\) gives

\[
\tan \theta = \frac{\sqrt{v^2 + u^2}}{u/r}
\]

By the definitions of sine and cosine, the numerator equals \(\sin \theta\) and the denominator equals \(\cos \theta\). As a result, these quotient properties are true.

### PROPERTIES: Quotient Properties for Tangent and Cotangent

\[
\tan \theta = \frac{\sin \theta}{\cos \theta} \quad \text{and} \quad \cot \theta = \frac{\cos \theta}{\sin \theta}
\]

The quotient properties allow you to construct the tangent and cotangent graphs from the sine and cosine.

**EXAMPLE 2**

On paper, sketch the graphs of \(y = \sin x\) and \(y = \cos x\). Use the quotient property to sketch the graph of \(y = \cot x\). Show the asymptotes and the points where the graph crosses the \(x\)-axis.

Draw the graphs of the sine and the cosine functions (dashed and solid, respectively) as in Figure 3-3c. Because \(\cot \theta = \frac{\cos \theta}{\sin \theta}\), show the asymptotes where \(\sin \theta = 0\), and show the \(x\)-intercepts where \(\cos \theta = 0\).

At \(\theta = 45^\circ\), and wherever else the graphs of the sine and the cosine functions intersect each other, \(\cot \theta\) will equal 1. Wherever sine and cosine are opposites of each other, \(\cot \theta\) will equal \(-1\). Mark these points as in Figure 3-3f. Then sketch the cotangent graph through the marked points, consistent with the asymptotes. The final graph is shown in Figure 3-3g.
Problem Set 3-3

Reading Analysis

From what you have read in this section, what do you consider to be the main idea? What feature do the graphs of the tangent, cotangent, secant, and cosecant functions have that sinusoids do not have, and why do they have this feature? What algebraic properties allow you to sketch the graph of the tangent or cotangent function from two sinusoids?

Quick Review

Problems Q1–Q7 refer to the equation \( y = 3 + 4 \cos (5\theta - 6^\circ) \).

Q1. The graph of the equation is called a —?—.
Q2. The amplitude is —?—.
Q3. The period is —?—.
Q4. The phase displacement with respect to \( y = \cos \theta \) is —?—.
Q5. The frequency is —?—.
Q6. The sinusoidal axis is at \( y = —?— \).
Q7. The lower bound is at \( y = —?— \).
Q8. What kind of function is \( y = x^5 \)?
Q9. What kind of function is \( y = 5 \)?
Q10. The “If . . .” part of the statement of a theorem is called the
   A. Conclusion  B. Hypothesis
   C. Converse     D. Inverse
   E. Contrapositive

1. Secant Function Problem
   a. Sketch two cycles of the parent cosine function \( y = \cos \theta \). Use the fact that \( \sec \theta = \frac{1}{\cos \theta} \) to sketch the graph of \( y = \sec \theta \).
   b. How can you locate the asymptotes in the secant graph by looking at the cosine graph? How does your graph compare with the secant graph in Figure 3-3b?
   c. Does the secant function have critical points? If so, find some of them. If not, explain why not.
   d. Does the secant function have points of inflection? If so, find some of them. If not, explain why not.

2. Tangent Function Problem
   a. Sketch two cycles of the parent function \( y = \cos \theta \) and two cycles of the parent function \( y = \sin \theta \) on the same axes.
   b. Explain how you can use the graphs in part a to locate the \( \theta \)-intercepts and the vertical asymptotes of the graph of \( y = \tan \theta \).
   c. Mark the asymptotes, intercepts, and other significant points on your sketch in part a. Then sketch the graph of \( y = \tan \theta \). How does the result compare with the tangent graph in Figure 3-3a?
   d. Does the tangent function have critical points? If so, find some of them. If not, explain why not.
   e. Does the tangent function have points of inflection? If so, find some of them. If not, explain why not.
3. Quotient Property for Tangent Problem: Plot these three graphs on the same screen on your grapher. Explain how the result confirms the quotient property for tangent.

\[ y_1 = \sin \theta \]
\[ y_2 = \cos \theta \]
\[ y_3 = y_1 / y_2 \]

4. Quotient Property for Cotangent Problem: On the same screen on your grapher, plot these three graphs. Explain how the result confirms the quotient property for cotangent.

\[ y_1 = \sin \theta \]
\[ y_2 = \cos \theta \]
\[ y_3 = y_2 / y_1 \]

5. Without referring to Figure 3-3a, quickly sketch the graphs of \( y = \tan \theta \) and \( y = \cot \theta \).

6. Without referring to Figure 3-3b, quickly sketch the graphs of \( y = \sec \theta \) and \( y = \csc \theta \).

7. Explain why the period of the functions \( y = \tan \theta \) and \( y = \cot \theta \) is only 180° instead of 360°, like the periods of the other four trigonometric functions.

8. Explain why it is meaningless to talk about the amplitude of the tangent, cotangent, secant, and cosecant functions.

9. What is the domain of the function \( y = \sec \theta \)? What is its range?

10. What is the domain of the function \( y = \tan \theta \)? What is its range?

For Problems 11–14, what are the dilation and translation caused by the constants in the equation? Plot the graph on your grapher and show that these transformations are correct.

11. \( y = 2 + 5 \tan (3\theta - 5^\circ) \)
12. \( y = -1 + 3 \cot (2\theta - 30^\circ) \)
13. \( y = 4 + 6 \sec \left( \frac{1}{2}(\theta + 50^\circ) \right) \)
14. \( y = 3 + 2 \csc 4(\theta + 10^\circ) \)

15. Rotating Lighthouse Beacon Problem:

Figure 3-3h shows a lighthouse located 500 m from the shore.

![Figure 3-3h](image)

A rotating light on top of the lighthouse sends out rays of light in opposite directions. As the beacon rotates, the ray at angle \( \theta \) makes a spot of light that moves along the shore. As \( \theta \) increases beyond 90°, the other ray makes the spot of light. Let \( D \) be the displacement of the spot of light from the point on the shore closest to the beacon, with the displacement positive to the right and negative to the left as you face the beacon from the shore.

a. Plot the graph of \( D \) as a function of \( \theta \).

Use a window with 0° to 360° for \( \theta \) and -2000 to 2000 for \( D \). Sketch the result.
b. Where does the spot of light hit the shore when \( \theta = 55^\circ \)? When \( \theta = 91^\circ \)?
c. What is the first positive value of \( \theta \) for which \( D \) equals 2000? For which \( D \) equals –1000?
d. Explain the physical significance of the asymptote in the graph at \( \theta = 90^\circ \).

16. Variation of Tangent and Secant Problem:
   Figure 3-3i shows the unit circle in a \( uv \)-coordinate system and a ray from the origin, \( O \), at an angle, \( \theta \), in standard position. The ray intersects the circle at point \( P \).
   A line is drawn tangent to the circle at \( P \), intersecting the \( u \)-axis at point \( A \) and the \( v \)-axis at point \( B \). A vertical segment from \( P \) intersects the \( u \)-axis at point \( C \), and a horizontal segment from \( P \) intersects the \( v \)-axis at point \( D \).

   Figure 3-3i

   a. Use the properties of similar triangles to explain why these segment lengths are equal to the six corresponding function values.
      \[
      \begin{align*}
      PA &= \tan \theta \\
      PB &= \cot \theta \\
      PC &= \sin \theta \\
      PD &= \cos \theta \\
      OA &= \sec \theta \\
      OB &= \csc \theta
      \end{align*}
      \]
   b. The angle between the ray and the \( v \)-axis is the complement of angle \( \theta \), that is, its measure is \( 90^\circ - \theta \). Show that in each case the cofunction of \( \theta \) is equal to the function of the complement of \( \theta \).
   c. Construct Figure 3-3i using dynamic geometry software such as The Geometer's Sketchpad, or use the Variation of Tangent and Secant Exploration at www.keymath.com/precalc. Observe what happens to the six function values as \( \theta \) changes. Describe how the sine and cosine vary as \( \theta \) is made larger or smaller. Based on the figure, explain why the tangent and secant become infinite as \( \theta \) approaches \( 90^\circ \) and why the cotangent and cosecant become infinite as \( \theta \) approaches \( 0^\circ \).

3-4 Radian Measure of Angles

With your calculator in degree mode, press \( \sin 60^\circ \). You get
\[
\sin 60^\circ = 0.866025403...
\]
Now change to radian mode and press \( \sin \left( \frac{\pi}{3} \right) \). You get the same answer!
\[
\sin \left( \frac{\pi}{3} \right) = 0.866025403...
\]
In this section you will learn what radians are and how to convert angle measures between radians and degrees. The radian measure of angles allows you to expand on the concept of trigonometric functions, as you’ll see in the next section. Through this expansion of trigonometric functions, you can model real-world phenomena in which independent variables represent distance, time, or any other quantity, not just an angle measure in degrees.
The degree as a unit of angular measure came from ancient mathematicians, probably Babylonians. It is assumed that they divided a revolution into 360 parts we call degrees because there were approximately 360 days in a year and they used the base-60 (sexagesimal) number system. There is another way to measure angles, called radian measure. This mathematically more natural unit of angular measure is derived by wrapping a number line around the unit circle (a circle of radius 1 unit) in a u-v-coordinate system, as in Figure 3-4a. Each point on the number line corresponds to a point on the perimeter of the circle.

If you draw rays from the origin to the points 1, 2, and 3 on the circle (right side of Figure 3-4a), the corresponding central angles have radian measures 1, 2, and 3, respectively.

But, you may ask, what happens if the same angle is in a larger circle? Would the same radian measure correspond to it? How would you calculate the radian measure in this case? Figures 3-4b and 3-4c answer these questions. Figure 3-4b shows an angle of measure 1, in radians, and the arcs it subtends (cuts off) on circles of radius 1 unit and x units. The arc subtended on the unit circle has length 1 unit. By the properties of similar geometric figures, the arc subtended on the circle of radius x has length x units. So 1 radian subtends an arc of length equal to the radius of the circle.
For any angle measure, the arc length and the radius are proportional \( \frac{a_1}{r_1} = \frac{a_2}{r_2} \), as shown in Figure 3-4c, and their quotient is a unitless number that uniquely corresponds to and describes the angle. So, in general, the radian measure of an angle equals the length of the subtended arc divided by the radius.

**DEFINITION: Radian Measure of an Angle**

\[
\text{radian measure} = \frac{\text{arc length}}{\text{radius}}
\]

For the work that follows, it is important to distinguish between the name of the angle and the measure of that angle. Measures of angle \( \theta \) will be written this way:

- \( \theta \) is the *name* of the angle.
- \( m^\circ(\theta) \) is the degree measure of angle \( \theta \).
- \( m^\pi(\theta) \) is the radian measure of angle \( \theta \).

Because the circumference of a circle is \( 2\pi r \) and because \( r \) for the unit circle is 1, the wrapped number line in Figure 3-4a divides the circle into \( 2\pi \) units (a little more than six parts). So there are \( 2\pi \) radians in a complete revolution. There are also 360° in a complete revolution. You can convert degrees to radians, or the other way around, by setting up these proportions:

\[
\frac{m^\circ(\theta)}{m^\pi(\theta)} = \frac{2\pi}{360^\circ} = \frac{\pi}{180^\circ} \quad \text{or} \quad \frac{m^\circ(\theta)}{m^\pi(\theta)} = \frac{360^\circ}{2\pi} = \frac{180^\circ}{\pi}
\]

Solving for \( m^\pi(\theta) \) and \( m^\circ(\theta) \), respectively, gives you

\[
m^\pi(\theta) = \frac{\pi}{180^\circ} m^\circ(\theta) \quad \text{and} \quad m^\circ(\theta) = \frac{180^\circ}{\pi} m^\pi(\theta)
\]

These equations lead to a procedure for accomplishing the objective of this section.

**PROCEDURE: Radian–Degree Conversion**

- To find the radian measure of \( \theta \), multiply the degree measure by \( \frac{\pi}{180^\circ} \).
- To find the degree measure of \( \theta \), multiply the radian measure by \( \frac{180^\circ}{\pi} \).
EXAMPLE 1

Convert 135° to radians.

Solution

In order to keep the units straight, write each quantity as a fraction with the proper units. If you have done the work correctly, certain units will cancel, leaving the proper units for the answer.

\[
m^\theta(\theta) = \frac{135 \text{ degrees}}{1} \cdot \frac{\pi \text{ radians}}{180 \text{ degrees}} = \frac{3}{4} \pi = 2.3561\ldots \text{ radians}
\]

Notes:

- If the exact value is called for, leave the answer as \(\frac{3}{4} \pi\). If not, you have the choice of writing the answer as a multiple of \(\pi\) or converting to a decimal.
- The procedure for canceling units used in Example 1 is called dimensional analysis. You will use this procedure throughout your study of mathematics.

EXAMPLE 2

Convert 5.73 radians to degrees.

Solution

\[
\frac{5.73 \text{ radians}}{1} \cdot \frac{180 \text{ degrees}}{\pi \text{ radians}} = 328.3048\ldots^\circ
\]

EXAMPLE 3

Find tan 3.7.

Solution

Unless the argument of a trigonometric function has the degree symbol, it is assumed to be a measure in radians. (That is why it has been important for you to include the degree symbol up till now.) Set your calculator to radian mode and enter tan 3.7.

\[\tan 3.7 = 0.6247\ldots\]

EXAMPLE 4

Find the radian measure and the degree measure of an angle whose sine is 0.3.

Solution

\[
\sin^{-1} 0.3 = 0.3046\ldots \text{ radian}
\]

Set your calculator to radian mode.

\[
\sin^{-1} 0.3 = 17.4576\ldots^\circ
\]

Set your calculator to degree mode.

To check whether these answers are in fact equivalent, you could convert one to the other.

\[
0.3046\ldots \text{ radian} \cdot \frac{180 \text{ degrees}}{\pi \text{ radians}} = 17.4576\ldots ^\circ
\]

Use the 0.3046… already in your calculator, without rounding off.

Radian Measures of Some Special Angles

It will help you later in calculus to be able to recall quickly the radian measures of certain special angles, such as those whose degree measures are multiples of 30° and 45°.
By the technique of Example 1,

\[ 30^\circ = \frac{\pi}{6} \text{ radian, or } \frac{1}{12} \text{ revolution} \]

\[ 45^\circ = \frac{\pi}{4} \text{ radian, or } \frac{1}{8} \text{ revolution} \]

If you remember these two, you can find others quickly by multiplication. For instance,

\[ 60^\circ = 2(\frac{\pi}{6}) = \frac{\pi}{3} \text{ radians, or } \frac{1}{6} \text{ revolution} \]

\[ 90^\circ = 3(\frac{\pi}{6}) \text{ or } 2(\frac{\pi}{4}) = \frac{\pi}{2} \text{ radians, or } \frac{1}{4} \text{ revolution} \]

\[ 180^\circ = 6(\frac{\pi}{6}) \text{ or } 4(\frac{\pi}{4}) = \pi \text{ radians, or } \frac{1}{2} \text{ revolution} \]

For 180°, you can simply remember that a full revolution is 2\pi radians, so half a revolution is \pi radians.

Figure 3-4d shows the radian measures of some special first-quadrant angles. Figure 3-4e shows radian measures of larger angles that are \frac{1}{8}, \frac{1}{4}, \frac{1}{2}, and 1 revolution. The box summarizes this information.

### PROPERTY: Radian Measures of Some Special Angles

<table>
<thead>
<tr>
<th>Degrees</th>
<th>Radians</th>
<th>Revolutions</th>
</tr>
</thead>
<tbody>
<tr>
<td>30°</td>
<td>(\frac{\pi}{6})</td>
<td>(\frac{1}{12})</td>
</tr>
<tr>
<td>45°</td>
<td>(\frac{\pi}{4})</td>
<td>(\frac{1}{8})</td>
</tr>
<tr>
<td>60°</td>
<td>(\frac{\pi}{3})</td>
<td>(\frac{1}{6})</td>
</tr>
<tr>
<td>90°</td>
<td>(\frac{\pi}{2})</td>
<td>(\frac{1}{4})</td>
</tr>
<tr>
<td>180°</td>
<td>(\pi)</td>
<td>(\frac{1}{2})</td>
</tr>
<tr>
<td>360°</td>
<td>(2\pi)</td>
<td>1</td>
</tr>
</tbody>
</table>
EXAMPLE 5  Find the exact value of $\sec \frac{\pi}{6}$.

**Solution**  
$\sec \frac{\pi}{6} = \sec 30^\circ = \frac{1}{\cos 30^\circ} = \frac{1}{\sqrt{3}/2} = \frac{2}{\sqrt{3}}$

Recall how to use the reference triangle to find the exact value of $\cos 30^\circ$.

**Problem Set 3-4**

**Reading Analysis**

From what you have read in this section, what do you consider to be the main idea? Is a radian large or small compared to a degree? How do you find the radian measure of an angle if you know its degree measure? How can you remember that there are $2\pi$ radians in a full revolution?

**Quick Review**

01. Sketch the graph of $y = \tan \theta$.

02. Sketch the graph of $y = \sec \theta$.

03. What is the first positive value of $\theta$ at which the graph of $y = \cot \theta$ has a vertical asymptote?

04. What is the first positive value of $\theta$ for which the graph of $\csc \theta = 0$?

05. What is the exact value of $\tan 60^\circ$?

06. What transformation of function $f$ is represented by $g(x) = 3f(x)$?

07. What transformation of function $f$ is represented by $h(x) = f(10x)$?

08. Write the general equation of a quadratic function.

09. $3^{2007} \div 3^{2008} = -7$?

10. The “then” part of the statement of a theorem is called the
    A. Converse  
    B. Inverse  
    C. Contrapositive  
    D. Conclusion  
    E. Hypothesis

Section 3-4: Radian Measure of Angles
2. Arc Length and Angle Problem: As a result of the definition of radian, you can calculate the arc length as the product of the angle in radians and the radius of the circle. Figure 3-4g shows arcs of three circles subtended by a central angle of 1.3 radians. The radii of the circles have lengths 1, 2, and 3 cm.

a. How long would the arc of the 1-cm circle be if you measured it with a flexible ruler?
b. Find how long the arcs are on the 2-cm circle and on the 3-cm circle using the properties of similar geometrical figures.
c. On a circle of radius \( r \) meters, how long would an arc be that is subtended by an angle of 1.3 radians?
d. How could you quickly find the length \( a \) of an arc of a circle of radius \( r \) meters that is subtended by a central angle of \( \theta \) radians?

Write a formula representing the arc length.

For Problems 3–10, find the exact radian measure of the angle (no decimals).

3. 60°
4. 45°
5. 30°
6. 180°
7. 120°
8. 450°
9. −225°
10. 1080°

For Problems 11–14, find the radian measure of the angle in decimal form.

11. 37°
12. 54°
13. 123°
14. 258°

For Problems 15–24, find the exact degree measure of the angle given in radians (no decimals). Use the most time-efficient method.

15. \( \frac{\pi}{4} \) radian
16. \( \frac{\pi}{3} \) radian
17. \( \frac{\pi}{6} \) radian
18. \( \frac{\pi}{9} \) radian
19. \( \frac{\pi}{12} \) radian
20. \( \frac{\pi}{8} \) radian
21. \( \frac{\pi}{10} \) radians
22. \( \pi \) radians
23. \( \frac{7\pi}{4} \) radians
24. \( \frac{5\pi}{4} \) radians

For Problems 25–30, find the degree measure in decimal form of the angle given in radians.

25. 0.34 radian
26. 0.62 radian
27. 1.26 radians
28. 1.57 radians
29. 1 radian
30. 3 radians

For Problems 31–34, find the function value (in decimal form) for the angle in radians.

31. sin 5
32. cos 2
33. tan (−2.3)
34. sin 1066

For Problems 35–38, find the radian measure (in decimal form) of the angle.

35. \( \sin^{-1} 0.3 \)
36. \( \tan^{-1} 5 \)
37. \( \cot^{-1} 3 \)
38. \( \csc^{-1} 1.001 \)

For Problems 39–44, find the exact value of the indicated function (no decimals). Note that because the degree sign is not used, the angle is assumed to be in radians.

39. \( \sin \frac{\pi}{4} \)
40. \( \cos \pi \)
41. \( \tan \frac{\pi}{6} \)
42. \( \cot \frac{\pi}{2} \)
43. \( \sec 2\pi \)
44. \( \csc \frac{\pi}{4} \)
For Problems 45–48, find the exact value of the expression (no decimals).

45. \(\sin \frac{\pi}{4} + 6 \cos \frac{\pi}{4}\)
46. \(\csc \frac{\pi}{4} \sin \frac{\pi}{4}\)
47. \(\cos^2 \frac{\pi}{4} + \sin^2 \frac{\pi}{4}\)
48. \(\tan^2 \frac{\pi}{4} - \sec^2 \frac{\pi}{4}\)

For Problems 49 and 50, write a particular equation for the sinusoid graphed.

For Problems 51 and 52, find the length of the side marked \(x\) in the right triangle.

For Problems 53 and 54, find the degree measure of angle \(\theta\) in the right triangle.

---

3-5 Circular Functions

In many real-world situations, the independent variable of a periodic function is time or distance, with no angle evident. For instance, the normal daily high temperature varies periodically with the day of the year. In this section you will learn about circular functions, periodic functions whose independent variable is a real number without any units. These functions, as you will see, are identical to trigonometric functions in every way except for their argument. Circular functions are more appropriate for real-world applications. They also have some advantages in later courses in calculus, for which this course is preparing you.

**OBJECTIVE**

Learn about the circular functions and their relationship to trigonometric functions.
Two cycles of the graph of the parent cosine function are completed in 720° (Figure 3-5a, left) or in $4\pi$ units (Figure 3-5a, right), because $4\pi$ radians correspond to two revolutions.

To see how the independent variable can represent a real number, imagine the $x$-axis from an $xy$-coordinate system lifted out and placed vertically tangent to the unit circle in a $uv$-coordinate system with its origin at the point $(u, v) = (1, 0)$, as on the left side in Figure 3-5b. Then wrap the $x$-axis around the unit circle. As shown on the right side in Figure 3-5b, $x = 1$ maps onto an angle of 1 radian, $x = 2$ maps onto 2 radians, $x = \pi$ maps onto $\pi$ radians, and so on.

The distance $x$ on the $x$-axis is equal to the arc length on the unit circle. This arc length is equal to the radian measure for the corresponding angle. Thus the functions $\sin x$ and $\cos x$ for a number $x$ on the $x$-axis are the same as the sine and cosine of an angle of $x$ radians.

Figure 3-5c shows an arc of length $x$ on the unit circle, with the corresponding angle. The arc is in standard position on the unit circle, with its initial point at $(1, 0)$ and its terminal point at $(u, v)$. The sine and cosine of $x$ are defined in the same way as for the trigonometric functions.

\[
\cos x = \frac{\text{horizontal coordinate}}{\text{radius}} = \frac{u}{1} = u
\]

\[
\sin x = \frac{\text{vertical coordinate}}{\text{radius}} = \frac{v}{1} = v
\]
The name *circular function* comes from the fact that $x$ equals the length of an arc on the unit circle. The other four circular functions are defined as ratios of sine and cosine.

**DEFINITION: Circular Functions**

If $(u, v)$ is the terminal point of an arc of length $x$ in standard position on the unit circle, then the *circular functions* of $x$ are defined as:

<table>
<thead>
<tr>
<th>Function</th>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sin x$</td>
<td>$v$</td>
</tr>
<tr>
<td>$\cos x$</td>
<td>$u$</td>
</tr>
<tr>
<td>$\tan x$</td>
<td>$\frac{\sin x}{\cos x}$</td>
</tr>
<tr>
<td>$\cot x$</td>
<td>$\frac{\cos x}{\sin x}$</td>
</tr>
<tr>
<td>$\sec x$</td>
<td>$\frac{1}{\cos x}$</td>
</tr>
<tr>
<td>$\csc x$</td>
<td>$\frac{1}{\sin x}$</td>
</tr>
</tbody>
</table>

Circular functions are equivalent to trigonometric functions in radians. This equivalency provides an opportunity to expand the concept of trigonometric functions. You have seen trigonometric functions first defined using the angles of a right triangle and later expanded to include all angles. From now on, the concept of trigonometric functions includes circular functions, and the functions can have both degrees and radians as arguments. The way the two kinds of trigonometric functions are distinguished is by their arguments. If the argument is measured in degrees, Greek letters represent them (for example, $\sin \theta$). If the argument is measured in radians, the functions are represented by letters from the Roman alphabet (for example, $\sin x$).

**EXAMPLE 1**

Plot the graph of $y = 4 \cos 5x$ on your grapher, in radian mode. Find the period graphically and algebraically. Compare your results.

**Solution**

Figure 3-5d shows the graph.

Tracing the graph, you find that the first high point beyond $x = 0$ is between $x = 1.25$ and $x = 1.3$. So graphically the period is between 1.25 and 1.3.

To find the period algebraically, recall that the 5 in the argument of the cosine function is the reciprocal of the horizontal dilation. The period of the parent cosine function is $2\pi$, because there are $2\pi$ radians in a complete revolution. Thus the period of the given function is

$$\frac{1}{5}(2\pi) = 0.4\pi = 1.2566\ldots$$

The answer found graphically is close to this exact answer.
**EXAMPLE 2**

Find a particular equation for the sinusoid function graphed in Figure 3-5e. Notice that the horizontal axis is labeled \( x \), not \( \theta \), indicating that the angle is measured in radians. Confirm your answer by plotting the equation on your grapher.

\[ y = C + A \cos B(x - D) \]

**Solution**

- Sinusoidal axis is at \( y = 3 \), so \( C = 3 \).
- Amplitude is 2, so \( A = 2 \).
- Period is 10.
- Dilation is \( \frac{10}{2\pi} \) or \( \frac{5}{\pi} \), so \( B = \frac{\pi}{3} \).
- Phase displacement is 1 (for \( y = \cos x \)), so \( D = 1 \).

Plotting this equation in radian mode confirms that it is correct.

**EXAMPLE 3**

Sketch the graph of \( y = \tan \frac{\pi}{6} x \).

**Solution**

In order to graph the function, you need to identify its period, the locations of its inflection points, and its asymptotes.

\[ \text{Period} = \frac{6}{\pi} \cdot \frac{\pi}{6} = 6 \]

Horizontal dilation is the reciprocal of \( \frac{\pi}{6} \); the period of the tangent is \( \pi \).

For this function, the points of inflection are also the \( x \)-intercepts, or the points where the value of the function equals zero. So

\[ \frac{\pi}{6} x = 0, \pm \pi, \pm 2\pi, \ldots \]

\[ x = 0, \pm 6, \pm 12, \ldots \]
Asymptotes are at values where the function is undefined. So
\[
\frac{\pi}{6}, \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots
\]
\[
x = -3, 3, 9, 15, \ldots
\]
Recall that halfway between a point of inflection and an asymptote the tangent equals 1 or -1. The graph in Figure 3-5f illustrates these features.

Problem Set 3-5

Reading Analysis
From what you have read in this section, what do you consider to be the main idea? As defined in this text, what are the differences and the similarities between a circular function and a trigonometric function? How do angle measures in radians link the circular functions to the trigonometric functions?

Quick Review
Q1. How many radians are in 180°?
Q2. How many degrees are in 2\pi radians?
Q3. How many degrees are in 1 radian?
Q4. How many radians are in 34°?
Q5. Find \(\sin 47°\).
Q6. Find \(\sin 47\).
Q7. Find the period of \(y = 3 + 4 \cos 5(\theta - 6°)\).
Q8. Find the upper bound for \(y\) for the sinusoid in Problem Q7.
Q9. How long does it take you to go 300 mi at an average speed of 60 mi/h?
Q10. Write 5% as a decimal.

For Problems 1–4, find the exact arc length on the unit circle subtended by the given angle (no decimals).
1. 30°
2. 60°
3. 90°
4. 45°

For Problems 5–8, find the exact degree measure of the angle that subtends the given arc length of the unit circle.
5. \(\frac{\pi}{4}\) units
6. \(\frac{\pi}{2}\) unit
7. \(\frac{\pi}{2}\) unit
8. \(\frac{\pi}{4}\) units

For Problems 9–12, find the exact arc length on the unit circle subtended by the given angle in radians.
9. \(\frac{\pi}{4}\) radians
10. \(\pi\) radians
11. 2 radians
12. 1.467 radians

For Problems 13–16, evaluate the circular function in decimal form.
13. \(\tan 1\)
14. \(\sin 2\)
15. \(\sec 3\)
16. \(\cot 4\)

For Problems 17–20, find the inverse circular function in decimal form.
17. \(\cos^{-1} 0.3\)
18. \(\tan^{-1} 1.4\)
19. \(\csc^{-1} 5\)
20. \(\sec^{-1} 9\)

For Problems 21–24, find the exact value of the circular function (no decimals).
21. \(\sin \frac{\pi}{6}\)
22. \(\cos \frac{\pi}{4}\)
23. \(\tan \frac{\pi}{4}\)
24. \(\csc \pi\)
For Problems 25–28, find the period, amplitude, phase displacement, and sinusoidal axis location. Use these features to sketch the graph. Confirm your graph by plotting the sinusoids on your grapher.

25. \( y = 3 + 2 \cos \frac{x}{2}(x - 4) \)
26. \( y = -4 + 5 \sin \frac{2\pi}{3}(x + 1) \)
27. \( y = 2 + 6 \sin \frac{2\pi}{3}(x + 1) \)
28. \( y = 5 + 4 \cos \frac{2\pi}{3}(x - 2) \)

For Problems 29–32, find the period, asymptotes, and critical points or points of inflection, then sketch the graph.

29. \( y = \cot \frac{2\pi}{3}x \)  
30. \( y = \tan 2\pi x \)
31. \( y = 2 + \sec x \)  
32. \( y = 3 \csc x \)

For Problems 33–42, find a particular equation for the circular function graphed.
41. For the sinusoid in Problem 41, find the value of $z$ at $t = 0.4$ on the graph. If the graph is extended to $t = 50$, is the point on the graph above or below the sinusoidal axis? How far above or below?

42. For the sinusoid in Problem 42, find the value of $E$ at $r = 1234$ on the graph. If the graph is extended to $r = 10,000$, is the point on the graph above or below the sinusoidal axis? How far above or below?

43. For the sinusoid in Problem 41, find the value of $z$ at $t = 0.4$ on the graph. If the graph is extended to $t = 50$, is the point on the graph above or below the sinusoidal axis? How far above or below?

44. For the sinusoid in Problem 42, find the value of $E$ at $r = 1234$ on the graph. If the graph is extended to $r = 10,000$, is the point on the graph above or below the sinusoidal axis? How far above or below?

45. **Sinusoid Translation Problem:** Figure 3-5g shows the graphs of $y = \cos x$ (dashed) and $y = \sin x$ (solid). Note that the graphs are congruent to each other (if superimposed, they coincide), differing only in horizontal translation.

   a. What translation would make the cosine graph coincide with the sine graph? Complete the equation: $\sin x = \cos (\ldots)$. 
   
   b. Let $y = \cos (x - \beta)$, What effect does this translation have on the cosine graph?

   c. Name a positive and a negative translation that would make the sine graph coincide with itself.
   
   d. Explain why $\sin (x - 2\pi n) = \sin x$ for any integer $n$. How is the $2\pi$ related to the sine function?

   e. Using dynamic geometry software such as The Geometer's Sketchpad, plot two sinusoids with different colors illustrating the concept of this problem, or use the Sinusoid Translation Exploration at www.keymath.com/precalc. One sinusoid should be $y = \cos x$ and the other $y = \cos (x - k)$, where $k$ is a slider or parameter with values between $-2\pi$ and $2\pi$. Describe what happens to the transformed graph as $k$ varies.

46. **Sinusoid Dilation Problem:** Figure 3-5h shows the unit circle in a $uv$-coordinate system with angles of measure $x$ and $2x$ radians. The $uv$-coordinate system is superimposed on an $xy$-coordinate system with sinusoids $y = \sin x$ (dashed) and an image graph $y = \sin 2x$ (solid).

   a. Explain why the value of $v$ for each angle is equal to the value of $y$ for the corresponding sinusoid.

   b. Create Figure 3-5h with dynamic geometry software such as Sketchpad, or go to www.keymath.com/precalc and use the Sinusoid Dilation Exploration. Show the whole unit circle, and extend the $x$-axis to $x = 7$. Use a slider or parameter to vary the value of $x$. Is the second angle measure double the first one as $x$ varies? Do the moving points on the two sinusoids have the same value of $x$?
c. Replace the 2 in $\sin 2x$ with a variable factor, $k$. Use a slider or parameter to vary $k$. What happens to the period of the (solid) image graph as $k$ increases? As $k$ decreases?

47. **Circular Function Comprehension Problem:**
For circular functions such as $\cos x$, the independent variable, $x$, represents the length of an arc of the unit circle. For other functions you have studied, such as the quadratic function $y = ax^2 + bx + c$, the independent variable, $x$, stands for a distance along a horizontal number line, the $x$-axis.

a. Explain how the concept of wrapping the $x$-axis around the unit circle links the two kinds of functions.

b. Explain how angle measures in radians link the circular functions to the trigonometric functions.

c. From Figure 3-5i it appears that $\sin x < x < \tan x$. Make a table of values to show numerically that this inequality is true even for values of $x$ very close to zero.

d. Construct Figure 3-5i with dynamic geometry software such as Sketchpad, or go to [www.keymath.com/precalc](http://www.keymath.com/precalc) and use the Inequality $\sin x < x < \tan x$ Exploration. On your sketch, display the values of $x$ and the ratios $(\sin x)/x$ and $(\tan x)/x$. What do you notice about the relative sizes of these values when angle $AOB$ is in the first quadrant? What value do the two ratios seem to approach as angle $AOB$ gets close to zero?

49. **Journal Problem:** Update your journal with things you have learned about the relationship between trigonometric functions and circular functions.

3-6 **Inverse Circular Relations:**
**Given $y$, Find $x$**

A major reason for finding the particular equation of a sinusoid is to use it to evaluate $y$ for a given $x$-value or to calculate $x$ when you are given $y$. Functions are used this way to make predictions in the real world. For instance, you can express the time of sunrise as a function of the day of the year. With this equation, you can predict the time of sunrise on a given day by simply evaluating the expression. Predicting the day(s)
on which the Sun rises at a given time is more complicated. In this section you will learn graphical, numerical, and algebraic ways to find \( x \) for a given \( y \)-value.

**OBJECTIVE**

Given the equation of a circular function or trigonometric function and a particular value of \( y \), find specified values of \( x \) or \( \theta \):
- Graphically
- Numerically
- Algebraically

### The Inverse Cosine Relation

The symbol \( \cos^{-1} 0.3 \) means the inverse cosine function evaluated at 0.3, a particular arc or angle whose cosine is 0.3. By calculator, in radian mode,

\[
\cos^{-1} 0.3 = 1.2661\
\]

The inverse cosine relation includes all arcs or angles whose cosine is a given number. The term that you’ll use in this text is arccosine, abbreviated arccos. So arccos 0.3 means any arc or angle whose cosine is 0.3, not just the function value. Figure 3-6a shows that both 1.2661... and \(-1.2661...\) have cosines equal to 0.3. So \(-1.2661...\) is also a value of arccos 0.3.

The general solution for the arccosine of a number is written this way:

\[
\arccos x = \pm \cos^{-1} x + 2\pi n
\]

where \( n \) stands for an integer. The \( \pm \) sign tells you that both the value from the calculator and its opposite are values of arccos 0.3. The \( 2\pi n \) tells you that any arc that is an integer number of revolutions added to these values is also a value of arccos 0.3. If \( n \) is a negative integer, a number of revolutions is being subtracted from these values. Note that there are infinitely many such values.

The arcsine and arctangent relations will be defined in Section 4-4 in connection with solving more general equations.

**DEFINITION: Arccosine, the Inverse Cosine Relation**

\[
\arccos x = \pm \cos^{-1} x + 2\pi n \quad \text{or} \quad \arccos x = \pm \cos^{-1} x + 360^\circ n,
\]

where \( n \) is an integer

Verbally: Inverse cosines come in opposite pairs with all their coterminals.

---

Radar speed guns use inverse relations to calculate the speed of a car from time measurements.
Note: The function value $\cos^{-1} x$ is called the **principal value** of the inverse cosine relation. This is the value the calculator is programmed to give you. In Section 4-6, you will learn why certain quadrants are picked for these inverse function values.

**Example 1**

Find the first five positive values of $\arccos(-0.3)$.

**Solution**

Assume that the inverse circular function is being asked for.

$$\arccos(-0.3) = \pm \cos^{-1}(-0.3) + 2\pi n$$

By calculator,

$$= \pm 1.8754... + 2\pi n$$

or

$$= 1.8754..., 1.8754... + 2\pi, 1.8754... + 4\pi$$

Use $\cos^{-1}(-0.3)$.

$$\pm 1.8754... + 2\pi, -1.8754... + 4\pi$$

Arrange in ascending order.

$$= 1.8754..., 8.1586..., 14.4418... or 4.4076..., 10.6908...$$

**Note:** Do not round the value of $\cos^{-1}(-0.3)$ before adding the multiples of $2\pi$. An efficient way to do this on your calculator is

- Press $\cos^{-1}(-0.3) =$, getting 1.8754....
- Press Ans + 2\pi =, getting 8.1586....
- Press Ans + 2\pi =, getting 14.4418.... Or just press = to repeat the step before.

$$= 1.8754..., -1.8754... + 4\pi$$

Press $-\cos^{-1}(-0.3) + 2\pi =$, getting 4.4076....

Press Ans + 2\pi =, getting 10.6908....

**Finding x When You Know y**

Figure 3-6b shows a sinusoid with a horizontal line drawn at $y = 5$. The horizontal line cuts the part of the sinusoid shown at six different points. Each point corresponds to a value of $x$ for which $y = 5$. The next examples show how to find the values of $x$ by three methods.

**Example 2**

Find **graphically** the six values of $x$ for which $y = 5$ for the sinusoid in Figure 3-6b.
Solution

On the graph, draw lines from the intersection points down to the x-axis (Figure 3-6b). The values are

\[ x \approx -4.5, -0.5, 8.5, 12.5, 21.5, 25.5 \]

\[ \text{EXAMPLE 3} \]

Find **numerically** the six values of \( x \) in Example 2. Show that the answers agree with those found graphically in Example 2.

**Solution**

\[ y_1 = 9 + 7 \cos \frac{2\pi}{13}(x - 4) \]

\[ y_2 = 5 \]

\[ x \approx 8.5084... \quad \text{and} \quad x \approx 12.4915... \]

\[ x \approx 8.5084... + 13(-1) = 4.4915... \]

\[ x \approx 12.4915... + 13(-1) = 8.4915... \]

\[ x \approx 8.5084... + 13(1) = 21.5084... \]

\[ x \approx 12.4915... + 13(1) = 25.4915... \]

These answers agree with the answers found graphically in Example 2.

Note that the \( \approx \) sign is used for answers found numerically because the solver or intersect feature on most calculators gives only approximate answers.

\[ \text{EXAMPLE 4} \]

Find **algebraically** (by calculation) the six values of \( x \) in Example 2. Show that the answers agree with those in Examples 2 and 3.

**Solution**

\[ 9 + 7 \cos \frac{2\pi}{13}(x - 4) = 5 \]

Set the two functions equal to each other.

\[ \cos \frac{2\pi}{13}(x - 4) = -\frac{4}{7} \]

Simplify the equation by isolating the cosine expression (start "peeling" constants away from \( x \)).

\[ \frac{2\pi}{13}(x - 4) = \arccos \left(-\frac{4}{7}\right) \]

Take the arccosine of both sides.

\[ x = 4 + \frac{13}{2\pi} \arccos \left(-\frac{4}{7}\right) \]

Rearrange the equation to isolate \( x \) (finish "peeling" constants away from \( x \)).

\[ x = 4 + \frac{13}{2\pi} \left(\cos^{-1}\left(-\frac{4}{7}\right) + 2\pi n\right) \]

Substitute for arccosine.

\[ x = 4 + \frac{13}{2\pi} \cos^{-1}\left(-\frac{4}{7}\right) + 13n \]

Distribute the \( \frac{13}{2\pi} \) over both terms.
21.5084…, 25.4915… Let \( n \) be 0, ±1, ±2.

These answers agree with the graphical and numerical solutions in Examples 2 and 3.

Notes:

- In the term 13\( n \), the 13 is the period. The 13\( n \) in the general solution for \( x \) means that you need to add multiples of the period to the values of \( x \) you find for the inverse function.
- You can put 8.5084… + 13\( n \) and \(-0.5084… + 13\( n \) into the y= menu of your grapher and make a table of values. For most graphers you will have to use \( x \) in place of \( n \).
- The algebraic solution gets all the values at once rather than one at a time numerically.

Problem Set 3-6

**Reading Analysis**

From what you have read in this section, what do you consider to be the main idea? Why does the arccosine of a number have more than one value while \( \cos^{-1} \) of that number has only one value? What do you have to do to the inverse cosine value you get on your calculator in order to find other values of arccosine? Explain the phrase “Inverse cosines come in opposite pairs with all their coterminals” that appears in the definition box for arccosine.

**Quick Review**

**Q1.** What is the period of the trigonometric function \( y = \cos 4x \)?

**Q2.** What is the period of the trigonometric function \( y = \cos 4\theta \)?

**Q3.** How many degrees are in \( \frac{\pi}{2} \) radian?

**Q4.** How many radians are in 45°?

**Q5.** Sketch the graph of \( y = \sin \theta \).

**Q6.** Sketch the graph of \( y = \csc \theta \).

**Q7.** Find the smaller acute angle in a right triangle with legs of lengths 3 mi and 7 mi.

**Q8.** \( x^2 + y^2 = 9 \) is the equation of a(n) —?—.

**Q9.** What is the general equation of an exponential function?

**Q10.** Functions that repeat themselves at regular intervals are called —?— functions.

For Problems 1–4, find the first five positive values of the inverse circular relation.

1. \( \arccos 0.9 \) 2. \( \arccos 0.4 \)
3. \( \arccos (-0.2) \) 4. \( \arccos (-0.5) \)

For the circular sinusoids graphed in Problems 5–10,

a. Estimate graphically the \( x \)-values shown for the indicated \( y \)-value.

b. Find a particular equation of the sinusoid.

c. Find the \( x \)-values in part a numerically, using the equation from part b.

d. Find the \( x \)-values in part a algebraically.

e. Find the first value of \( x \) greater than 100 for which \( y \) = the given \( y \)-value.
5. \( y = 6 \)

6. \( y = 5 \)

7. \( y = -1 \)

8. \( y = -2 \)

9. \( y = 1.5 \)

10. \( y = -4 \)

For the trigonometric sinusoids graphed in Problems 11 and 12,

a. Estimate graphically the first three positive values of \( \theta \) for the indicated \( y \)-value.

b. Find a particular equation for the sinusoid.

c. Find the \( \theta \)-values in part a numerically, using the equation from part b.

d. Find the \( \theta \)-values in part a algebraically.

11. \( y = 3 \)

12. \( y = 5 \)

13. Figure 3-6c shows the graph of the parent cosine function \( y = \cos x \).

a. Find algebraically the six values of \( x \) shown on the graph for which \( \cos x = -0.9 \).

b. Find algebraically the first value of \( x \) greater than 200 for which \( \cos x = -0.9 \).

Section 3-6: Inverse Circular Relations: Given \( y \), Find \( x \)
3-7 Sinusoidal Functions as Mathematical Models

A chemotherapy treatment destroys red blood cells along with cancer cells. The red cell count goes down for a while and then comes back up again. If a treatment is taken every three weeks, then the red cell count resembles a periodic function of time (Figure 3-7a). If such a function is regular enough, you can use a sinusoidal function as a mathematical model.

In this section you’ll start with a verbal description of a periodic phenomenon, interpret it graphically, find an algebraic equation from the graph, and use the equation to calculate numerical answers.

**OBJECTIVE**

Given a verbal description of a periodic phenomenon, write an equation using the sine or cosine function and use the equation as a mathematical model to make predictions and interpretations about the real world.

**EXAMPLE 1**

Waterwheel Problem: Suppose that the waterwheel in Figure 3-7b rotates at 6 revolutions per minute (rev/min). Two seconds after you start a stopwatch, point $P$ on the rim of the wheel is at its greatest height, $d = 13$ ft, above the surface of the water. The center of the waterwheel is 6 ft above the surface.

a. Sketch the graph of $d$ as a function of time $t$, in seconds, since you started the stopwatch.

b. Assuming that $d$ is a sinusoidal function of $t$, write a particular equation. Confirm by graphing that your equation gives the graph you sketched in part a.

c. How high above or below the water’s surface will point $P$ be at time $t = 17.5$ s? At that time, will it be going up or down?

d. At what positive time $t$ was point $P$ first emerging from the water?
Solution

a. From what’s given, you can tell the location of the sinusoidal axis, the “high” and “low” points, and the period.

Sketch the sinusoidal axis at \( d = 6 \) as shown in Figure 3-7c.

Sketch the upper bound at \( d = 6 + 7 = 13 \) and the lower bound at \( d = 6 - 7 = -1 \).

Sketch a high point at \( t = 2 \).

Because the waterwheel rotates at 6 rev/min, the period is \( \frac{60}{6} = 10 \) s. Mark the next high point at \( t = 2 + 10 \), or 12.

Mark a low point halfway between the two high points, and mark the points of inflection on the sinusoidal axis halfway between each consecutive high and low.

Sketch the graph through the critical points and the points of inflection. Figure 3-7c shows the finished sketch.

b. \( d = C + A \cos B(t - D) \) Write the general equation. Use \( d \) and \( t \) for the variables.

From the graph, \( C = 6 \) and \( A = 7 \).

\( D = 2 \) Cosine starts a cycle at a high point.

Horizontal dilation: \( \frac{10}{2\pi} = \frac{5}{\pi} \) The period of this sinusoid is 10; the period of the circular cosine function is \( 2\pi \).

\( B = \frac{\pi}{5} \) \( B \) is the reciprocal of the horizontal dilation.

\( \therefore d = 6 + 7 \cos \frac{\pi}{5}(t - 2) \) Write the particular equation.

Plotting on your grapher confirms that the equation is correct (Figure 3-7d).

c. Set the window on your grapher to include 17.5. Then trace or scroll to this point (Figure 3-7d). From the graph, \( d = -0.6573... \), or \( -0.7 \) ft, and is going up.

d. Point \( P \) is either submerging into or emerging from the water when \( d = 0 \).

At the first zero for positive \( t \)-values, shown in Figure 3-7d, the point is going into the water. At the next zero, the point is emerging. Using the intersect, zeros, or solver feature of your grapher, you’ll find that the point is at

\[ t = 7.8611... \approx 7.9 \text{ s} \]

If you go to \( \text{www.keymath.com/precalc} \), you can view the Waterwheel Exploration for a dynamic view of the waterwheel and the graph of \( d \) as a function of \( t \).

Note that it is usually easier to use the cosine function for these problems, because its graph starts a cycle at a high point.
Problem Set 3-7

Reading Analysis

From what you have read in this section, what do you consider to be the main idea? What is the first step in solving a sinusoidal model problem that takes it out of the real world and puts it into the mathematical world? After you have taken this step, how does your work in this chapter allow you to answer questions about the real-world situation?

Quick Review

Problems Q1–Q8 concern the circular function $y = 4 + 5 \cos \frac{\pi}{6}(x - 7)$.

Q1. The amplitude is —?—.
Q2. The period is —?—.
Q3. The frequency is —?—.
Q4. The sinusoidal axis is at $y = —?—$.
Q5. The phase displacement with respect to the parent cosine function is —?—.
Q6. The upper bound is at $y = —?—$.
Q7. If $x = 9$, then $y = —?—$.
Q8. The first three positive x-values at which low points occur are —?—, —?—, and —?—.
Q9. Two values of $x = \arccos 0.5$ are —?— and —?—.
Q10. If $y = 5 - 3^x$, adding 2 to the value of x multiplies the value of y by —?—.

c. Find a particular equation for distance as a function of time.

d. How far above the surface was the point when Mark’s stopwatch read 17 s?

e. What is the first positive value of $t$ at which the point was at the water’s surface? At that time, was the point going into or coming out of the water? How can you tell?
f. “Mark Twain” is a pen name used by Samuel Clemens. What is the origin of that pen name? Give the source of your information.

2. Fox Population Problem: Naturalists find that populations of some kinds of predatory animals vary periodically with time. Assume that the population of foxes in a certain forest varies sinusoidally with time. Records started being kept at time $t = 0$ yr. A minimum number of 200 foxes appeared at $t = 2.9$ yr. The next maximum, 800 foxes, occurred at $t = 5.1$ yr.

a. Sketch the graph of this sinusoid.
b. Find a particular equation expressing the number of foxes as a function of time.
c. Predict the fox population when $t = 7, 8, 9,$ and $10$ yr.
d. Foxes are declared an endangered species when their population drops below 300. Between what two nonnegative values of $t$ did the foxes first become endangered?
3. **Bouncing Spring Problem:** A weight attached to the end of a long spring is bouncing up and down (Figure 3-7e). As it bounces, its distance from the floor varies sinusoidally with time. You start a stopwatch. When the stopwatch reads 0.3 s, the weight first reaches a high point 60 cm above the floor. The next low point, 40 cm above the floor, occurs at 1.8 s.

   a. Sketch the graph of this sinusoidal function.
   b. Find a particular equation for distance from the floor as a function of time.
   c. What is the distance from the floor when the stopwatch reads 17.2 s?
   d. What was the distance from the floor when you started the stopwatch?
   e. What is the first positive value of time when the weight is 59 cm above the floor?

4. **Rope Swing Problem:** Zoey is at summer camp. One day she is swinging on a rope tied to a tree branch, going back and forth alternately over land and water. Nathan starts a stopwatch.

   When \( x = 2 \) s, Zoey is at one end of her swing, at a distance \( y = -23 \) ft from the riverbank (see Figure 3-7f). When \( x = 5 \) s, she is at the other end of her swing, at a distance \( y = 17 \) ft from the riverbank. Assume that while she is swinging, \( y \) varies sinusoidally with \( x \).

   a. Sketch the graph of \( y \) versus \( x \) and write a particular equation.
   b. Find \( y \) when \( x = 13.2 \) s. Was Zoey over land or over water at this time?
   c. Find the first positive time when Zoey was directly over the riverbank (\( y = 0 \)).
   d. Zoey lets go of the rope and splashes into the water. What is the value of \( y \) for the end of the rope when it comes to rest? What part of the mathematical model tells you this?

5. **Roller Coaster Problem:** A theme park is building a portion of a roller coaster track in the shape of a sinusoid (Figure 3-7g). You have been hired to calculate the lengths of the horizontal and vertical support beams.

   a. The high and low points of the track are separated by 50 m horizontally and 30 m vertically. The low point is 3 m below the ground. Let \( y \) be the distance (in meters) a point on the track is above the ground. Let \( x \) be the horizontal distance (in meters) a point on the track is from the high point. Find a particular equation for \( y \) as a function of \( x \).
b. The vertical support beams are spaced 2 m apart, starting at the high point and ending just before the track goes below the ground. Make a table of values of the lengths of the beams.

c. The horizontal beams are spaced 2 m apart, starting at ground level and ending just below the high point. Make a table of values of horizontal beam lengths.

d. The builder must know how much support beam material to order. In the most time-efficient way, find the total length of the vertical beams and the total length of the horizontal beams.

The valley to the left is filled with water to a depth of 50 m, and the top of the mountain is 150 m above the water level. You set up an x-axis at water level and a y-axis 200 m to the right of the deepest part of the water. The top of the mountain is at $x = 400$ m.

a. Find a particular equation expressing $y$ for points on the surface of the mountain as a function of $x$.

b. Show algebraically that the sinusoid in part a contains the origin, $(0, 0)$.

c. The treasure is located beneath the surface at the point $(130, 40)$, as shown in Figure 3-7h. Which would be a shorter way to dig to the treasure, a horizontal tunnel or a vertical tunnel? Show your work.

6. Buried Treasure Problem: Suppose you seek a treasure that is buried in the side of a mountain. The mountain range has a sinusoidal vertical cross section (Figure 3-7h).

The valley to the left is filled with water to a depth of 50 m, and the top of the mountain is 150 m above the water level. You set up an x-axis at water level and a y-axis 200 m to the right of the deepest part of the water. The top of the mountain is at $x = 400$ m.

a. What is the period of a sunspot cycle?

7. Sunspot Problem: For several hundred years, astronomers have kept track of the number of solar flares, or “sunspots,” that occur on the surface of the Sun. The number of sunspots in a given year varies periodically, from a minimum of about 10 per year to a maximum of about 110 per year. Between 1750 and 1948, there were exactly 18 complete cycles.

a. What is the period of a sunspot cycle?
b. Assume that the number of sunspots per year is a sinusoidal function of time and that a maximum occurred in 1948. Find a particular equation expressing the number of sunspots per year as a function of the year.

c. How many sunspots will there be in the year 2020? This year?

d. What is the first year after 2020 in which there will be a maximum number of sunspots?

e. Find out how closely the sunspot cycle resembles a sinusoid by looking on the Internet or in another reference.

8. Tide Problem: Suppose that you are on the beach at Port Aransas, Texas, on August 2. At 2:00 p.m., at high tide, you find that the depth of the water at the end of a pier is 1.5 m. At 7:30 p.m., at low tide, the depth of the water is 1.1 m. Assume that the depth varies sinusoidally with time.

a. Find a particular equation expressing depth as a function of the time that has elapsed since 12:00 midnight at the beginning of August 2.

b. Use your mathematical model to predict the depth of the water at 5:00 p.m. on August 3.

c. At what time does the first low tide occur on August 3?

d. What is the earliest time on August 3 that the water depth will be 1.27 m?

e. A high tide occurs because the Moon is pulling the water away from Earth slightly, making the water a bit deeper at a given point. How do you explain the fact that there are two high tides each day at most places on Earth? Provide the source of your information.

9. Shock Felt Round the World Problem: Suppose that one day all 200+ million people in the United States climb up on tables. At time $t = 0$, they all jump off. The resulting shock wave starts the earth vibrating at its fundamental period, 54 min. The surface first moves down from its normal position and then moves up an equal distance above its normal position (Figure 3-7i). Assume that the amplitude is 50 m.

a. Sketch the graph of the displacement of the surface from its normal position as a function of time elapsed since the people jumped.

b. At what time will the surface be farthest above its normal position?

c. Write a particular equation expressing displacement above normal position as a function of time elapsed since the jump.

d. What is the displacement at $t = 21$?

e. What are the first three positive times at which the displacement is $D = 75$ m?

10. Island Problem: Ona Nyland owns an island several hundred feet from the shore of a lake. Figure 3-7j shows a vertical cross section through the shore, lake, and island. The island was formed millions of years ago by stresses that caused the earth's surface to warp into the sinusoidal pattern shown. The highest point on the shore is at $x = 150$ ft. From measurements on and near the shore...
(solid part of the graph), topographers find that an equation of the sinusoid is

\[ y = -70 + 100 \cos \frac{\pi}{600}(x + 150) \]

where \( x \) and \( y \) are in feet. Ona consults you to make predictions about the rest of the graph (dotted).

\[ \text{Figure 3-7j} \]

a. What is the highest the island rises above the water level in the lake? How far from the \( y \)-axis is this high point? Show how you got your answers.

b. What is the deepest the sinusoid goes below the water level in the lake? How far from the \( y \)-axis is this low point? Show how you got your answers.

c. Over the centuries silt has filled the bottom of the lake so that the water is only 40 ft deep. That is, the silt line is at \( y = 40 \) ft. Plot the graph. Use a friendly window for \( x \) and a window with a suitable range for \( y \). Then find graphically the interval of \( x \)-values between which Ona would expect to find silt if she goes scuba diving in the lake.

d. If Ona drills an offshore well at \( x = 700 \) ft, through how much silt would she drill before she reaches the sinusoid? Show how you got your answer.

e. The sinusoid appears to go through the origin. Does it actually do so, or does it just miss? Justify your answer.

f. Find algebraically the interval of \( x \)-values between which the island is at or above the water level. How wide is the island, from the water on one side to the water on the other?

11. **Pebble in the Tire Problem:** As you stop your car at a traffic light, a pebble becomes wedged between the tire treads. When you start moving again, the distance between the pebble and the pavement varies sinusoidally with the distance you have gone. The period is the circumference of the tire. Assume that the diameter of the tire is 24 in.

a. Sketch the graph of this sinusoidal function.

b. Find a particular equation of the function.

(c It is possible to get an equation with zero phase displacement.)

c. What is the pebble’s distance from the pavement when you have gone 15 in.?

d. What are the first two distances you have gone when the pebble is 11 in. from the pavement?

12. **Oil Well Problem:** Figure 3-7k shows a vertical cross section through a piece of land. The \( y \)-axis is drawn coming out of the ground at the fence bordering land owned by your boss, Earl Wells. Earl owns the land to the left of the fence and is interested in acquiring land on the other side to drill a new oil well. Geologists have found an oil-bearing formation below Earl’s land that they believe to be sinusoidal in shape. At \( x = 100 \) ft, the top surface of the formation is at its deepest, \( y = 2500 \) ft. A quarter-cycle closer to the fence, at \( x = 65 \) ft, the top surface is only 2000 ft deep. The first 700 ft of land beyond the fence is inaccessible. Earl wants to drill at the first convenient site beyond \( x = 700 \) ft.

\[ \text{Figure 3-7k} \]
a. Find a particular equation expressing $y$ as a function of $x$.
b. Plot the graph on your grapher. Use a window with an $x =$ range of about $[-100, 900]$. Describe how the graph confirms that your equation is correct.
c. Find graphically the first interval of $x$-values in the available land for which the top surface of the formation is no more than 1600 ft deep.
d. Find algebraically the values of $x$ at the ends of the interval in part c. Show your work.
e. Suppose that the original measurements were slightly inaccurate and that the value of $y$ shown at 65 ft instead is at $x = 64$. Would this fact make much difference in the answer to part c? Use the most time-efficient method to arrive at your answer. Explain what you did.

13. **Sound Wave Problem**: The hum you hear on some radios when they are not tuned to a station is a sound wave of 60 cycles per second.

![Sound wave image]

Bats navigate and communicate using ultrasonic sounds with frequencies of 20–100 kilohertz (kHz), which are undetectable by the human ear. A kilohertz is 1000 cycles per second.

a. Is 60 cycles per second the period, or is it the frequency? If it is the period, find the frequency. If it is the frequency, find the period.
b. The wavelength of a sound wave is defined as the distance the wave travels in a time interval equal to one period. If sound travels at 1100 ft/s, find the wavelength of the 60-cycle-per-second hum.
c. The lowest musical note the human ear can hear is about 16 cycles per second. In order to play such a note, a pipe on an organ must be exactly half as long as the wavelength. What length organ pipe would be needed to generate a 16-cycle-per-second note?

14. **Sunrise Project**: Assume that the time of sunrise varies sinusoidally with the day of the year. Let $t$ be the time of sunrise. Let $d$ be the day of the year, starting with $d = 1$ on January 1.

a. On the Internet or from an almanac, find for your location the time of sunrise on the longest day of the year, June 21, and on the shortest day of the year, December 21. If you choose, you can use the data for San Antonio, 5:34 a.m. and 7:24 a.m., CST, respectively. The phase displacement for cosine will be the value of $d$ at which the Sun rises the latest. Use the information to find a particular equation expressing time of sunrise as a function of the day number.
b. Calculate the time of sunrise today at the location used for the equation in part a. Compare the answer to your data source.
c. What is the time of sunrise on your birthday, taking daylight saving time into account if necessary?
d. What is the first day of the year on which the Sun rises at 6:07 a.m. in the location in part a?
e. In the northern hemisphere, Earth moves faster in wintertime, when it is closer to the
Sun, and slower in summertime, when it is farther from the Sun. As a result, the actual high point of the sinusoid occurs later than predicted, and the actual low point occurs earlier than predicted (Figure 3-7l). A representation of the actual graph can be plotted by putting in a phase displacement that varies. See if you can duplicate the graph in Figure 3-7l on your grapher. Is the modified graph a better fit for the actual sunrise data for the location in part a?

Figure 3-7l

15. Variable Amplitude Pendulum Project: If there were no friction, the displacement of a pendulum from its rest position would be a sinusoidal function of time,

\[ y = A \cos Bt \]

To account for friction, assume that the amplitude \( A \) decreases exponentially with time,

\[ A = a \cdot b^t \]

Make a pendulum by tying a weight to a string hung from the ceiling or some other convenient place (see Figure 3-7m).

Find its period by measuring the time for 10 swings and dividing by 10. Record the amplitude when you first start the pendulum, and measure it again after 30 s. From these measurements, find the constants \( a, b, \) and \( B \) and write a particular equation expressing the position of the pendulum as a function of time. Test your equation by using it to predict the displacement of the pendulum at time \( t = 10 \) s and seeing if the pendulum really is where you predicted it to be at that time. Write an entry in your journal describing this experiment and your results.

\[ \begin{array}{c|c}
  t & d \\
  0 & 10 \\
  1 & 42.2 \\
  2 & 40.9 \\
  3 & 41.2 \\
  4 & 23.3 \\
  5 & 32.2 \\
  6 & 32.1 \\
  7 & 24.3 \\
\end{array} \]

3-8 Rotary Motion

When you ride a merry-go-round, you go faster when you sit nearer the outside. As the merry-go-round rotates through a certain angle, you travel farther in the same amount of time when you sit closer to the outside (Figure 3-8a).
However, all points on the merry-go-round turn through the same number of degrees per unit of time. So there are two different kinds of speed, or velocity, associated with a point on a rotating object. The angular velocity is the number of degrees per unit of time, and the linear velocity is the distance per unit of time.

**OBJECTIVE**

Given information about a rotating object or connected rotating objects, find linear and angular velocities of points on the objects.

To reduce rotary motion to familiar algebraic terms, certain symbols are usually used for radius, arc length, angle measure, linear velocity, angular velocity, and time (Figure 3-8b). They are:

- **r** Radius from the center of rotation to the point in question
- **a** Number of units of arc length through which the point moves
- **θ** Angle through which the point rotates (usually in radians, but not always)
- **v** Linear velocity, in distance per time
- **ω** Angular velocity (often in radians per unit of time; Greek “omega”)
- **t** Length of time to rotate through a particular angle θ

![Figure 3-8b](image-url)
These definitions relate the variables.

**DEFINITIONS: Angular Velocity and Linear Velocity**

The angular velocity, \( \omega \), of a point on a rotating object is the number of degrees (radians, revolutions, and so on) through which the point turns per unit of time.

The linear velocity, \( v \), of a point on a rotating object is the distance the point travels along its circular path per unit of time.

Algebraically:

\[
\omega = \frac{\theta}{t} \quad \text{and} \quad v = \frac{a}{t}
\]

Properties of linear and angular velocity help you accomplish this section’s objective. First, by the definition of radians, the length of an arc of a circle is equal to the radius multiplied by the radian measure of the central angle. In physics, \( \theta \) is used for angles, even if the angle is measured in radians. Because you might study rotary motion elsewhere, you’ll see the same notation here.

\[
a = r \theta \quad \theta \text{ must be in radians.}
\]

\[
\frac{a}{t} = \frac{r \theta}{t} = r \cdot \frac{\theta}{t} \quad \text{Divide both sides of the equation by time.}
\]

By definition, the left side equals the linear velocity, \( v \), and the right side is \( r \) multiplied by the angular velocity, \( \omega \). So you can write the equation

\[
v = r \omega \quad \omega \text{ must be in radians per unit of time.}
\]

**PROPERTIES: Linear Velocity and Angular Velocity**

If \( \theta \) is in radians and \( \omega \) is in radians per unit of time, then

\[
a = r \theta
\]

\[
v = r \omega
\]

---

**Analysis of a Single Rotating Object**

**EXAMPLE 1**

An old LP (“long play”) record, as in Figure 3–8c, rotates at 33 1/3 rev/min.

a. Find the angular velocity in radians per second.

b. Find the angular and linear velocities of the record (per second) at the point at which the needle is located when it is just starting to play, 14.5 cm from the center.

c. Find the angular and linear velocities (per second) at the center of the turntable.
Solution

a. The 33 1/2 rev/min is already an angular velocity because it is a number of revolutions (angle) per unit of time. All you need to do is change to the desired units. For this purpose, it is helpful to use dimensional analysis. There are $2\pi$ radians in one revolution and 60 seconds in 1 minute. Write the conversion factors this way:

$$\omega = \frac{33 \frac{1}{2} \text{ rev}}{\text{min}} \cdot \frac{2\pi \text{ rad}}{\text{rev}} \cdot \frac{1 \text{ min}}{60 \text{ s}} = \frac{33 \frac{1}{2} \pi}{3.4906\ldots} \approx 3.49 \text{ rad/s}$$

Notice that the revolutions and minutes cancel, leaving radians in the numerator and seconds in the denominator.

b. All points on the same rotating object have the same angular velocity. So the point 14.5 cm from the center is also rotating at $\omega = \frac{1}{2\pi}$ radians per second. The computation of linear velocity is

$$v = r\omega = \frac{14.5 \text{ cm} \cdot \frac{1}{2\pi} \text{ rad}}{\text{s}} = \frac{50.6145\ldots}{50.6} \approx 50.6 \text{ cm/s}$$

Note that for the purpose of dimensional analysis, the radius has the units "cm/rad." A point 14.5 cm from the center moves 14.5 cm along the arc for each radian the record rotates.

c. The turntable and record rotate as a single object. So all points on the turntable have the same angular velocity as the record, even the point that is the center of the turntable. The radius to the center is, of course, zero. So

$$\omega = \frac{1}{2\pi} \approx 3.49 \text{ rad/s}$$

$$v = r\omega = (0)\frac{1}{2\pi} \approx 0 \text{ cm/s}$$

Interestingly, the center of a rotating object has zero linear velocity, but it still rotates with the same angular velocity as all other points on the object.

Analysis of Connected Rotating Objects

Figure 3–8d shows the back wheel of a bicycle. A small sprocket is connected to the axle of the wheel. This sprocket is connected by a chain to the large sprocket to which the pedals are attached. So there are several rotating objects whose motions are related to each other. Example 2 shows you how to analyze the motion.

EXAMPLE 2

A cyclist turns the pedals of her bicycle (Figure 3-8d) at 8 rad/s. The front sprocket has diameter 20 cm and is connected by the chain to the back sprocket, which has diameter 6 cm. The rear wheel has radius 35 cm and is connected to the back sprocket.

a. What is the angular velocity of the front sprocket?

b. What is the linear velocity of points on the chain?
c. What is the linear velocity of points on the rim of the back sprocket?
d. What is the angular velocity of the center of the back sprocket?
e. How fast is the bicycle going in kilometers per hour?

Solution

a. \( \omega = 8 \text{ rad/s} \)

Because the pedals and the front sprocket are connected at their axles, they rotate as one object. All points on the same rotating object have the same angular velocity.

b. \( v = r \omega = \frac{10 \text{ cm}}{\text{rad}} \cdot \frac{8 \text{ rad}}{s} = 80 \text{ cm/s} \)

The linear velocity of points on the chain is the same as the linear velocity of points on the rim of the front sprocket. The radius of the front sprocket is 20/2, or 10 cm.

c. \( v = 80 \text{ cm/s} \)

The back sprocket’s rim has the same linear velocity as the chain and the front sprocket’s rim.

d. \( v = r \omega \Rightarrow \omega = \frac{v}{r} = \frac{80 \text{ cm}}{3 \text{ cm}} = \frac{262}{3} \text{ rad/s} \)

The angular velocity is the same at every point on the same rotating object, even at the center. So the angular velocity at the center of the back sprocket is the same as at the rim. You can calculate this angular velocity using the equation \( v = r \omega \). The radius is 3 cm, half the diameter.

e. \( v = r \omega = \frac{35 \text{ cm}}{\text{rad}} \cdot \frac{262}{3} \text{ rad/s} = \frac{3,600 \text{ s}}{100,000 \text{ cm}} = 33.6 \text{ km/h} \)

The wheel is connected by an axle to the back sprocket, so it rotates with the same angular velocity as the sprocket. Unless the wheel is skidding, the speed the bicycle goes is the same as the linear velocity of points on the rim of the wheel. You can calculate this linear velocity using the equation \( v = r \omega \).

From Example 2, you can draw some general conclusions about rotating objects connected either at their rims or by an axle.

**CONCLUSIONS: Connected Rotating Objects**

1. Two rotating objects connected by an axle have the same **angular velocity**.
2. Two rotating objects connected at their rims have the same **linear velocity** at their rims.
Problem Set

Reading Analysis

From what you have read in this section, what do you consider to be the main idea? Give a real-world example involving rotary motion. What is the difference between linear velocity and angular velocity? Explain why it is possible for one type of velocity to equal zero when the other does not equal zero.

Quick Review

Q1. A runner goes 1000 m in 200 s. What is her average speed?
Q2. A skater rotates 3000 deg in 4 s. How fast is he rotating?
Q3. If one value of \( \theta = \arccos x \) is 37\(^\circ\), then another value of \( \theta \) in \([0^\circ, 360^\circ]\) is —?—.
Q4. If one value of \( y = \arccos x \) is 1.2 radians, then the first negative value of \( y \) is —?—.
Q5. What is the period of the function \( y = 7 + 4 \cos 2(x - 5) \)?
Q6. What transformation of function \( f \) is \( g(x) = f(0.2x) \)?
Q7. Sketch a right triangle with hypotenuse 8 cm and one leg 4 cm. How long is the other leg?
Q8. What are the measures of the angles of the triangle in Problem Q7?
Q9. Factor: \( x^2 - 11x + 10 \)
Q10. Find the next term in the geometric sequence 3, 6, . . . .

1. Shot Put Problem: An athlete spins around in the shot put event to propel the shot. In order for the shot to land where he wants, it must leave his hand at a speed of 60 ft/s. Assume that the shot is 4 ft from his center of rotation.
   a. How many radians per second must he rotate to achieve his objective?
   b. How many revolutions per minute must he rotate?

2. Ship’s Propeller Problem: The propeller on a freighter has a radius of about 4 ft (Figure 3-8e). At full speed, the propeller turns at 150 rev/min.
   a. What is the angular velocity of the propeller in radians per second at the tip of the blades? At the center of the propeller?
   b. What is the linear velocity in feet per second at the tip of the blades? At the center of the propeller?

3. Lawn Mower Blade Problem: The blade on a rotary lawn mower is 19 in. long. The cutting edges begin 6 in. from the center of the blade (Figure 3-8f). In order for a lawn mower blade to cut grass, it must strike the grass at a speed of at least 900 in./s.
   a. If you want the innermost part of the cutting edge to cut grass, how many radians per second must the blade turn? How many revolutions per minute is this?
   b. What is the linear velocity of the outermost tip of the blade while it is turning as in part a?
   c. If the outermost tip of the blade strikes a stone while it is turning as in part a, how fast could the stone be propelled from the mower? How many miles per hour is this?
4. **Bicycle Problem:** Rhoda rides a racing bike at a speed of 50.4 km/h. The wheels have diameter 70 cm.
   a. What is the linear velocity of the points farthest out on the wheels?
   b. Find the angular velocity of the wheels in radians per second.
   c. Find the angular velocity of the wheels in revolutions per minute.

5. **Dust Problem:** A speck of dust is sitting 4 cm from the center of a turntable. Phoebe spins the turntable through an angle of $120^\circ$.
   a. Through how many radians does the speck of dust turn?
   b. What distance does it travel?
   c. If Phoebe rotates the turntable $120^\circ$ in 0.5 s, what is the dust speck’s angular velocity? What is its linear velocity?

6. **Seesaw Problem:** Stan and his older brother Ben play on a seesaw. Stan sits at a point 8 ft from the pivot. On the other side of the seesaw, Ben, who is heavier, sits just 5 ft from the pivot. As Ben goes up and Stan goes down, the seesaw rotates through an angle of $37^\circ$ in 0.7 s.
   a. What are Ben’s angular velocity in radians per second and linear velocity in feet per second?
   b. What are Stan’s angular and linear velocities?

7. **Figure Skating Problem:** Ima N. Aspin goes figure skating. She goes into a spin with her arms outstretched, making four complete revolutions in 6 s.
   a. How fast is she rotating in revolutions per second?
   b. Find Ima’s angular velocity in radians per second.
   c. Ima’s outstretched fingertips are 70 cm from the central axis of her body (around which she rotates). What is the linear velocity of her fingertips?
   d. As Ima spins there are points on her body that have zero linear velocity. Where are these points? What is her angular velocity at these points?
   e. Ima pulls her arms in close to her body, just 15 cm from her axis of rotation. As a result, her angular velocity increases to 10 rad/s. Are her fingertips going faster or slower than they were in part c? Justify your answer.

8. **Paper Towel Problem:** In 0.4 s, Dwayne pulls from the roll three paper towels with total length 45 cm (Figure 3-8g).
   a. How fast is he pulling the paper towels?
   b. The roll of towels has diameter 14 cm. What is the linear velocity of a point on the outside of the roll?
   c. What is the angular velocity of a point on the outside of the roll?
   d. How many revolutions per minute is the roll of towels spinning?
   e. The next day Dwayne pulls the last few towels off the roll. He pulls with the same linear velocity as before, but this time the roll’s diameter is only 4 cm. What is the angular velocity now?
9. **Pulley Problem:** Two pulleys are connected by a pulley belt (Figure 3-8h).

   ![Figure 3-8h](image)

   a. The small pulley has diameter 10 cm and rotates at 100 rev/min. Find its angular velocity in rad/s.
   b. Find the linear velocity of a point on the rim of the 10-cm pulley.
   c. Find the linear velocity of a point on the belt connecting the two pulleys.
   d. Find the linear velocity of a point on the rim of the large pulley, which has diameter 30 cm.
   e. Find the angular velocity of a point on the rim of the 30-cm pulley.
   f. Find the angular and linear velocities of a point at the center of the 30-cm pulley.

10. **Gear Problem:** A gear with diameter 30 cm is revolving at 45 rev/min. It drives a smaller gear that has diameter 8 cm (similar to Figure 3-8i).

   ![Figure 3-8i](image)

   a. How fast is the large gear turning in radians per minute?
   b. What is the linear velocity of the teeth on the large gear?
   c. What is the linear velocity of the teeth on the small gear?
   d. How fast is the small gear turning in radians per minute?
   e. How fast is the small gear turning in revolutions per minute?
   f. If you double an angular velocity by using gears, what is the ratio of the diameters of the gears? Which gear does the driving, the large gear or the small gear?

11. **Tractor Problem:** The rear wheels of a tractor (Figure 3-8j) are 4 ft in diameter and are turning at 20 rev/min.

   ![Figure 3-8j](image)

   a. How fast is the tractor going in feet per second? How fast is this in miles per hour?
   b. The front wheels have a diameter of only 1.8 ft. How fast are the tread points moving in feet per second? Is this an angular velocity or a linear velocity?
   c. How fast in revolutions per minute are the front wheels turning? Is this an angular velocity or a linear velocity?

12. **Wheel and Grindstone Problem:** A waterwheel with diameter 12 ft turns at 0.3 rad/s (Figure 3-8k).

   ![Figure 3-8k](image)

   a. What is the linear velocity of points on the rim of the waterwheel?
   b. The waterwheel is connected by an axle to a grindstone with diameter 3 ft. What is the angular velocity of points on the rim of the grindstone?
   c. What is the fastest velocity of any point on the grindstone? Where are these points?
13. **Three Gear Problem:** Three gears are connected as depicted schematically (without showing their teeth) in Figure 3-8l.

   a. Gear 1 rotates at 300 rev/min. Its radius is 8 in. What is its angular velocity in radians per second?
   
   b. Gear 2 is attached to the same axle as Gear 1 but has radius 2 in. What is its angular velocity?
   
   c. What is the linear velocity at a point on the teeth of Gear 2?
   
   d. Gear 3 is driven by Gear 2. What is the linear velocity of the teeth on Gear 3?
   
   e. Gear 3 has radius 18 in. What is the angular velocity of its teeth?
   
   f. What are the linear and angular velocities at the center of Gear 3?

   ![Figure 3-8l](image)

14. **Truck Problem:** In the 1930s, some trucks used a chain to transmit power from the engine to the wheels (Figure 3-8m). Suppose the drive sprocket had diameter 6 in., the wheel sprocket had diameter 20 in., and the drive sprocket rotated at 300 rev/min.

   a. Find the angular velocity of the drive sprocket in radians per second.
   
   b. Find the linear velocity of the wheel sprocket in inches per minute.
   
   c. Find the angular velocity of the wheel in radians per minute.
   
   d. If the wheel has diameter 38 in., find the speed the truck is going, to the nearest mile per hour.

   ![Figure 3-8m](image)

15. **Marching Band Formation Problem:** Suppose a marching band executes a formation in which some members march in a circle 50 ft in diameter and others in a circle 20 ft in diameter. The band members in the small circle march in such a way that they mesh with the members in the big circle without bumping into each other. Figure 3-8n shows the formation. The members in the big circle march at a normal pace of 5 ft/s.

   a. What is the angular velocity of the big circle in radians per second?
   
   b. What is the angular velocity of the big circle in revolutions per minute?
   
   c. Which is the same about the two circles, their linear or their angular velocities at the rims?

   ![Figure 3-8n](image)
d. What is the angular velocity of the small circle?
e. How many times faster does the small circle revolve? How can you find this factor using only the two diameters?

16. **Four Pulley Problem:** Four pulleys are connected to each other as shown in Figure 3-8o. Pulley 1 is driven by a motor at an angular velocity of 120 rev/min. It is connected by a belt to Pulley 2. Pulley 3 is on the same axle as Pulley 2. It is connected by another belt to Pulley 4. The dimensions of the pulleys are

- Pulley 1: radius = 10 cm
- Pulley 2: radius = 2 cm
- Pulley 3: diameter = 24 cm
- Pulley 4: radius = 3 cm

![Four Pulley Problem Diagram]

a. What is the angular velocity of Pulley 1 in radians per minute?
b. What is the linear velocity of the rim of Pulley 17?
c. Find the linear and angular velocities of the rim of Pulley 2.
d. Find the linear and angular velocities of the rim of Pulley 3.
e. Find the linear and angular velocities of the rim of Pulley 4.
f. Find the linear and angular velocities of the center of Pulley 4.
g. Find the angular velocity of Pulley 4 in revolutions per minute.
h. How many times faster than Pulley 1 is Pulley 4 rotating? How can you find this factor simply from the radii of the four pulleys?

17. **Gear Train Problem:** When something that rotates fast, like a car’s engine, drives something that rotates slower, like the car’s wheels, a gear train is used. In Figure 3-8p, Gear 1 is rotating at 2700 rev/min. The teeth on Gear 1 drive Gear 2, which is connected by an axle to Gear 3. The teeth on Gear 3 drive Gear 4. The sizes of the gears are

- Gear 1: radius = 2 cm
- Gear 2: radius = 15 cm
- Gear 3: radius = 3 cm
- Gear 4: radius = 18 cm

![Gear Train Problem Diagram]

a. What is the angular velocity of Gear 1 in radians per second?
b. Find the linear and angular velocities of the teeth on the rim of Gear 2.
c. Find the linear and angular velocities of the teeth on the rim of Gear 3.
d. Find the linear and angular velocities of the teeth on the rim of Gear 4.
e. Find the linear and angular velocities at the center of Gear 4.
f. Find the angular velocity of Gear 4 in revolutions per minute.
g. The **reduction ratio** is the ratio of the angular velocity of the fastest gear to the angular velocity of the slowest gear. What is the reduction ratio for the gear train in Figure 3-8p? Calculate this reduction ratio without working parts a–f of this problem.
3-9  Chapter Review and Test

In this chapter you learned how to graph trigonometric functions. The sine and cosine functions are continuous sinusoids, while other trigonometric functions are discontinuous, having vertical asymptotes at regular intervals. You also learned about circular functions, which you can use to model real-world phenomena mathematically, and you learned how radians provide a link between these circular functions and the trigonometric functions. Radians also provide a way to calculate linear and angular velocity in rotary motion problems.

Review Problems

R0. Update your journal with what you have learned since the last entry. Include things such as
• The one most important thing you have learned as a result of studying this chapter
• The graphs of the six trigonometric functions
• How the transformations of sinusoidal graphs relate to function transformations in Chapter 1
• How the circular and trigonometric functions are related
• Why circular functions usually are more appropriate as mathematical models than are trigonometric functions

R1. a. Sketch the graph of a sinusoid. On the graph, show the difference in meaning between a cycle and a period. Show the amplitude, the phase displacement, and the sinusoidal axis.

b. In the equation \( y = 3 + 4 \cos 5(\theta - 10^\circ) \), what name is given to the quantity \( 5(\theta - 10^\circ) \)?

R2. a. Without using your grapher, show that you understand the effects of the constants in a sinusoidal equation by sketching the graph of \( y = 3 + 4 \cos 5(\theta - 10^\circ) \). Give the amplitude, period, sinusoidal axis location, and phase displacement.

b. Using the cosine function, find a particular equation of the sinusoid in Figure 3-9a. Find another particular equation using the sine function. Show that the equations are equivalent to each other by plotting them on the same screen. What do you observe about the two graphs?

Figure 3-9a
c. A quarter-cycle of a sinusoid is shown in Figure 3-9b. Find a particular equation of the sinusoid.

![Figure 3-9b](image)

d. At what value of $\theta$ shown in Figure 3-9b does the graph have a point of inflection? At what point does the graph have a critical point?

e. Find the frequency of the sinusoid in Figure 3-9b.

R3. a. Sketch the graph of $y = \tan \theta$.

b. Explain why the period of the tangent function is $180^\circ$ rather than $360^\circ$ like sine and cosine.

c. Plot the graph of $y = \sec \theta$ on your grapher. Explain how you did this.

d. Use the relationship between sine and cosecant to explain why the cosecant function has vertical asymptotes at $\theta = 0^\circ$, $180^\circ$, $360^\circ$, . . .

e. Explain why the graph of the cosecant function has high and low points but no points of inflection. Explain why the graph of the cotangent function has points of inflection but no high or low points.

f. For the function $y = 2 + 0.4 \cot \frac{\pi}{4}(\theta - 40^\circ)$, give the vertical and horizontal dilations and the vertical and horizontal translations. Then plot the graph to confirm that your answers are correct. What is the period of this function? Why is it not meaningful to talk about its amplitude?

R4. a. How many radians are in $30^\circ$? In $45^\circ$? In $60^\circ$?

Give the answers exactly, in terms of $\pi$.

b. How many degrees are in an angle of 2 radians? Write the answer as a decimal.

c. Find $\cos 3$ and $\cos 3^\circ$.

d. Find the radian measure of $\cos^{-1} 0.8$ and $\csc^{-1} 2$.

e. How long is the arc of a circle subtended by a central angle of 1 radian if the radius of the circle is 17 units?

R5. a. Draw the unit circle in a $uv$-coordinate system. In this coordinate system, draw an $x$-axis vertically with its origin at the point $(u, v) = (1, 0)$. Show where the points $x = 1, 2,$ and 3 units map onto the unit circle as the $x$-axis is wrapped around it.

b. How long is the arc of the unit circle subtended by a central angle of $60^\circ$? Of $2.3$ radians?

c. Find $\sin 2^\circ$ and $\sin 2$.

d. Find the value of the inverse trigonometric function $\cos^{-1} 0.6$.

e. Find the exact values (no decimals) of the circular functions $\cos \frac{\pi}{4}$, $\sec \frac{\pi}{4}$, and $\tan \frac{\pi}{4}$.

f. Sketch the graphs of the parent circular functions $y = \cos x$ and $y = \sin x$.

g. Explain how to find the period of the circular function $y = 3 + 4 \sin \frac{\pi}{6}(x - 2)$ from the constants in the equation. Sketch the graph. Confirm by plotting on your grapher that your sketch is correct.

h. Find a particular equation of the circular function sinusoid for which a half-cycle is shown in Figure 3-9c.

![Figure 3-9c](image)

R6. a. Find the general solution of the inverse circular relation $\arccos 0.8$.

b. Find the first three positive values of the inverse circular relation $\arccos 0.8$.

c. Find the least value of $\arccos 0.1$ that is greater than 100.
d. For the sinusoid in Figure 3-9d, find the four values of x shown for which \( y = 2 \)
   - Graphically, to one decimal place
   - Numerically, by finding the particular equation and plotting the graph
   - Algebraically, using the particular equation

e. What is the next positive value of \( x \) for which \( y = 2 \), beyond the last positive value shown in Figure 3-9d?

R7. Porpoising Problem: Assume that you are aboard a research submarine doing submerged training exercises in the Pacific Ocean. At time \( t = 0 \), you start porpoising (going alternately deeper and shallower). At time \( t = 4 \) min you are at your deepest, \( y = -1000 \) m. At time \( t = 9 \) min you next reach your shallowest, \( y = +200 \) m. Assume that \( y \) varies sinusoidally with time.

a. Sketch the graph of \( y \) versus \( t \).
b. Write an equation expressing \( y \) as a function of \( t \).
c. Your submarine can’t communicate with ships on the surface when it is deeper than \( y = -300 \) m. At time \( t = 0 \), could your submarine communicate? How did you arrive at your answer?
d. Between what two nonnegative times is your submarine first unable to communicate?

R8. Clock Problem: The “second” hand on a clock rotates through an angle of 120° in 20 s.
a. What is its angular velocity in degrees per second?
b. What is its angular velocity in radians per second?
c. How far does a point on the tip of the hand, 11 cm from the axle, move in 20 s? What is the linear velocity of the tip of the hand? How can you calculate this linear velocity quickly from the radius and the angular velocity?

Three Wheel Problem: Figure 3-9e shows Wheel 1 with radius 15 cm, turning with an angular velocity of 50 rad/s. It is connected by a belt to Wheel 2, with radius 3 cm. Wheel 3, with radius 25 cm, is connected to the same axle as Wheel 2.

d. Find the linear velocity of points on the belt connecting Wheel 1 to Wheel 2.
e. Find the linear velocity of points on the rim of Wheel 2.
f. Find the linear velocity of a point at the center of Wheel 2.
g. Find the angular velocity of Wheel 2.
h. Find the angular velocity of Wheel 3.
i. Find the linear velocity of points on the rim of Wheel 3.
j. If Wheel 3 is touching the ground, how fast (in kilometers per hour) would the vehicle connected to the wheel be moving?
Concept Problems

C1. Pump Jack Problem: An oil well pump jack is shown in Figure 3-9f. As the motor turns, the walking beam rocks back and forth, pulling the rod out of the well and letting it go back into the well. The connection between the rod and the walking beam is a steel cable that wraps around the cathead. The distance $d$ from the ground to point $P$, where the cable connects to the rod, varies periodically with time.

a. As the walking beam rocks, the angle $\theta$ it makes with the ground varies sinusoidally with time. The angle goes from a minimum of $-0.2$ radian to a maximum of $0.2$ radian. How many degrees correspond to this range of angle ($\theta$)?

b. The radius of the circular arc on the cathead is 8 ft. What arc length on the cathead corresponds to the range of angles in part a?

c. The distance, $d$, between the cable-to-rod connector and the ground varies sinusoidally with time. What is the amplitude of the sinusoid?

d. Suppose that the pump is started at time $t = 0$ s. One second later, $P$ is at its highest point above the ground. $P$ is at its next low point 2.5 s after that. When the walking beam is horizontal, point $P$ is 7 ft above the ground. Sketch the graph of this sinusoid.

e. Find a particular equation expressing $d$ as a function of $t$.

f. How far above the ground is $P$ at $t = 9$?

g. How long does $P$ stay more than 7.5 ft above the ground on each cycle?

h. True or false? “The angle is always the independent variable in a periodic function.”

C2. Inverse Circular Relation Graphs: In this problem you’ll investigate the graphs of the inverse sine and inverse cosine functions and the general inverse sine and cosine relations from which they come.

a. On your grapher, plot the inverse circular function $y = \sin^{-1} x$. Use a window with an $x$-range of about $[-10, 10]$ that includes $x = 1$ and $x = -1$ as grid points. Use the same scales on both the $x$- and $y$-axes. Sketch the result.

b. The graph in part a is only for the inverse sine function. You can plot the entire inverse sine relation, $y = \arcsin x$, by putting your grapher in parametric mode. In this mode, both $x$ and $y$ are functions of a third variable, usually $t$. Enter the parametric equations this way:

$$x = \sin t$$

$$y = t$$

Plot the graph, using a window with a $t$-range the same as the $x$-range in part a. Sketch the graph.

c. Describe how the graphs in part a and part b are related to each other.

d. Explain algebraically how the parametric functions in part b and the function $y = \sin^{-1} x$ are related.
e. Find a way to plot the ordinary sine function, \( y = \sin x \), on the same screen, as in part b. Use a different style for this graph so that you can distinguish it from the other one. The result should look like the graphs in Figure 3-9g.

f. How are the two graphs in Figure 3-9g related to each other? Find a geometric transformation of the sine graph that gives the arcsine graph.

g. Explain why the arcsine graph in Figure 3-9g is not a function graph but the principal value of the inverse sine you plotted in part a is a function graph.

h. Using the same scales as in part b, plot the graphs of the cosine function, \( y = \cos x \), and the inverse cosine relation. Sketch the result. Do the two graphs have the same relationship as those in Figure 3-9g?

i. Repeat part h for the inverse tangent function.

j. Write an entry in your journal telling what you have learned from this problem.

C3. Merry-Go-Round Problem: A merry-go-round rotates at a constant angular velocity while rings of seats rotate at a different (but constant) angular velocity (Figure 3-9h). Suppose that the seats rotate at 30 rev/min counterclockwise while the merry-go-round is rotating at 12 rev/min counterclockwise.

a. Find your linear velocity, in feet per second, due to the combined rotations of the seats and the merry-go-round when your seat is

- Farthest from the center of the merry-go-round.
- Closest to the center of the merry-go-round.

b. In what direction are you actually moving when your seat is closest to the center of the merry-go-round?

c. As your seat turns, your distance from the fence varies sinusoidally with time. As the merry-go-round turns, the axis of this sinusoid also varies sinusoidally with time, but with a different period and amplitude. Suppose that at time \( t = 0 \) s your seat is at its farthest distance from the fence, 23 ft. Write an equation expressing your distance from the fence as a function of time, \( t \).

d. Plot the graph of the function in part c. Sketch the result.

e. Use the answers above to explain why many people don’t feel well after riding on this type of ride.


**Chapter Test**

**PART 1: No calculators allowed (T1–T9)**

T1. Figure 3-9i shows an x-axis drawn tangent to the unit circle in a uv-coordinate system. On a copy of this figure, show approximately where the point \(x = 2.3\) maps onto the unit circle when the x-axis is wrapped around the circle.

T2. Sketch an angle of 2.3 radians on the copy of Figure 3-9i.

T3. What are the steps needed to find a decimal approximation of the degree measure of an angle of 2.3 radians? In what quadrant would this angle terminate?

T4. Give the exact number of radians in 120° (no decimals).

T5. Give the exact number of degrees in \(\frac{\pi}{2}\) radian (no decimals).

T6. Give the period, amplitude, vertical translation, and phase displacement of this circular function:

\[ f(x) = 3 + 4 \cos \frac{\pi}{5}(x - 1) \]

T7. Sketch at least two cycles of the sinusoid in Problem T6.

T8. An object rotates with angular velocity \(\omega = 3\) rad/s. What is the linear velocity of a point 20 cm from the axis of rotation?

T9. A gear with radius 5 in. rotates so that its teeth have linear velocity 40 in./s. Its teeth mesh with a larger gear with radius 10 in. What is the linear velocity of the teeth on the larger gear?

**PART 2: Graphing calculators allowed (T10–T24)**

T10. A long pendulum hangs from the ceiling. As it swings back and forth, its distance from the wall varies sinusoidally with time. At time \(x = 1\) s it is at its closest point, \(y = 50\) cm. Three seconds later it is at its farthest point, \(y = 160\) cm. Sketch the graph.

T11. Figure 3-9j shows a half-cycle of a circular function sinusoid. Find a particular equation of this sinusoid.

T12. Find a time at which the water is deepest. How deep is it at that time?

For Problems T12–T18, Figure 3-9k shows the depth of the water at a point near the shore as it varies due to the tides. A particular equation relating \(d\), in feet, to \(t\), in hours after midnight on a given day, is

\[ d = 3 + 2 \cos \frac{\pi}{56}(t - 4) \]

T13. Find a time at which the water is deepest. How deep is it at that time?
T13. After the time you found in Problem T12, when is the water next at its shallowest? How deep is it at that time?

T14. What does \( t \) equal at 3:00 p.m.? How deep is the water at that time?

T15. Plot the graph of the sinusoid in Figure 3-9k on your grapher. Use a window with an \( x \)-range (actually, \( t \)) of about \([0, 50]\) and an appropriate window for \( y \) (actually, \( d \)).

T16. By tracing your graph in Problem T15, find, approximately, the first interval of nonnegative times for which the water is less than 4.5 ft deep.

T17. Set your grapher’s table mode to begin at the later time from Problem T16, and set the table increment at 0.01. Find to the nearest 0.01 h the latest time at which the water is still less than 4.5 ft deep.

T18. Solve algebraically for the first positive time at which the water is exactly 4.5 ft deep.

_Bicycle Problem:_ For Problems T19–T23, Anna Racer is riding her bike. She turns the pedals at 120 rev/min. The dimensions of the bicycle are shown in Figure 3-9l.

T19. What is the angular velocity of the pedals in radians per second?

T20. What is the linear velocity of the chain in centimeters per second?

T21. What is the angular velocity of the back wheel?

T22. How fast is Anna’s bike going, in kilometers per hour?

T23. The pedals are 24 cm from the axis of the large sprocket. Sketch a graph showing the distance of Anna’s right foot from the pavement as a function of the number of seconds since her foot was at a high point. Show the upper and lower bounds, the sinusoidal axis, and the location of the next three high points.

T24. What did you learn as a result of taking this test that you did not know before?