But serving up an action, suggesting the dynamic in the static, has become a hobby of mine . . . The "flowing" on that motionless plane holds my attention to such a degree that my preference is to try and make it into a cycle.

M. C. ESCHER

Waterfall, M. C. Escher, 1961
©2002 Cordon Art B.V.–Baarn–Holland. All rights reserved.

Escher has cleverly used right angles to form his artwork known as Waterfall. The picture contains three uses of the impossible tribar created by British mathematician Roger Penrose (b 1931) in 1954. In 1934 Swedish artist Oscar Reutersvard (b 1915), “father of impossible figures,” had created an impossible tribar that consisted of a triangular arrangement of cubes.

The shapes topping the towers in Escher’s work are, on the left, a compound of three cubes and, on the right, a stellation of the rhombic dodecahedron.

[Ask] “What impossible things do you see?” [Water seems to be traveling up an incline, yet it is running a mill wheel.] “Which surfaces appear to be horizontal? Vertical? Sloped? There are three impossible tribars in the picture; where are they?” [They all have flowing water along two sides; twice one of the bars is replaced by the waterfall, and once one bar is replaced by a group of four columns.]
LESSON 9.1

CHAPTER 9 The Pythagorean Theorem

The puzzle in this investigation is intended to help you recall the Pythagorean Theorem. It uses a dissection, which means you will cut apart one or more geometric figures and make the pieces fit into another figure.

Step 1
Construct a scalene right triangle in the middle of your paper. Label the hypotenuse \( c \) and the legs \( a \) and \( b \). Construct a square on each side of the triangle.

Step 2
To locate the center of the square on the longer leg, draw its diagonals. Label the center \( O \).

Step 3
Through point \( O \), construct line \( j \) perpendicular to the hypotenuse and line \( k \) perpendicular to line \( j \). Line \( k \) is parallel to the hypotenuse. Lines \( j \) and \( k \) divide the square on the longer leg into four parts.

Step 4
Cut out the square on the shorter leg and the four parts of the square on the longer leg. Arrange them to exactly cover the square on the hypotenuse.

The Theorem of Pythagoras

In a right triangle, the side opposite the right angle is called the hypotenuse. The other two sides are called legs. In the figure at right, \( a \) and \( b \) represent the lengths of the legs, and \( c \) represents the length of the hypotenuse.

There is a special relationship between the lengths of the legs and the length of the hypotenuse. This relationship is known today as the Pythagorean Theorem.

Investigation

The Three Sides of a Right Triangle

The puzzle in this investigation is intended to help you recall the Pythagorean Theorem. It uses a dissection, which means you will cut apart one or more geometric figures and make the pieces fit into another figure.

You will need
- scissors
- a compass
- a straightedge
- patty paper

Step 1
Construct a scalene right triangle in the middle of your paper. Label the hypotenuse \( c \) and the legs \( a \) and \( b \). Construct a square on each side of the triangle.

Step 2
To locate the center of the square on the longer leg, draw its diagonals. Label the center \( O \).

Step 3
Through point \( O \), construct line \( j \) perpendicular to the hypotenuse and line \( k \) perpendicular to line \( j \). Line \( k \) is parallel to the hypotenuse. Lines \( j \) and \( k \) divide the square on the longer leg into four parts.

Step 4
Cut out the square on the shorter leg and the four parts of the square on the longer leg. Arrange them to exactly cover the square on the hypotenuse.

Many students may already know the Pythagorean Theorem as \( a^2 + b^2 = c^2 \). In this lesson they review what the letters stand for and discover proofs showing why the relationship holds for all right triangles.

INTRODUCTION

Direct students’ attention to Improving Your Visual Thinking Skills on page 454. Ask what they can conclude about right triangles, and help them state the Pythagorean Theorem using areas of squares and the terms hypotenuse and legs.

Guiding the Investigation

One step
Hand out a copy of the Pythagorean Theorem worksheet to each group. Challenge students to cut up one or both of the smaller squares and assemble the pieces on top of the largest square. As you circulate, you might remind students of the problem-solving technique of trying a special case first—here, an isosceles right triangle. As needed, point out that good pieces might be formed if they draw lines through the smaller squares parallel to edges of the largest square.

[Language] A dissection is the result of separating something into pieces.

Step 1
Using the Dissection of Squares worksheets or the Sketchpad demonstration will speed the investigation, but the use of many different triangles drawn by the students strengthens the inductive conclusion.

The constructions are quicker with patty paper than with compass and straightedge. It is also easy to create several examples using geometry software.

Step 2
As needed, remind students that the legs are the sides other than the hypotenuse, so the "longer leg" is not the hypotenuse. Suggest that students minimize clutter by making these diagonals very light or by drawing only the portion near the center of the square.

Step 4
Ask students to take care in drawing and cutting out pieces so they will fit together well. Students may want to tape the pieces together.
The Pythagorean Theorem

In a right triangle, the sum of the squares of the lengths of the legs equals the square of the length of the hypotenuse. If $a$ and $b$ are the lengths of the legs, and $c$ is the length of the hypotenuse, then $a^2 + b^2 = c^2$.

History

Pythagoras of Samos (ca. 569–475 B.C.E.), depicted in this statue, is often described as “the first pure mathematician.” Samos was a principal commercial center of Greece and is located on the island of Samos in the Aegean Sea. The ancient town of Samos now lies in ruins, as shown in the photo at right. Mysteriously, none of Pythagoras’s writings still exist, and we know very little about his life. He founded a mathematical society in Croton, in what is now Italy, whose members discovered irrational numbers and the five regular solids. They proved what is now called the Pythagorean Theorem, although it was discovered and used 1000 years earlier by the Chinese and Babylonians. Some math historians believe that the ancient Egyptians also used a special case of this property to construct right angles.

A theorem is a conjecture that has been proved. Demonstrations like the one in the investigation are the first step toward proving the Pythagorean Theorem.

Believe it or not, there are more than 200 proofs of the Pythagorean Theorem. Elisha Scott Loomis’s Pythagorean Proposition, first published in 1927, contains original proofs by Pythagoras, Euclid, and even Leonardo da Vinci and U. S. President James Garfield. One well-known proof of the Pythagorean Theorem is included below. You will complete another proof as an exercise.

Paragraph Proof: The Pythagorean Theorem

You need to show that $a^2 + b^2$ equals $c^2$ for the right triangles in the figure at left. The area of the entire square is $(a + b)^2$ or $a^2 + 2ab + b^2$. The area of any triangle is $\frac{1}{2}ab$, so the sum of the areas of the four triangles is $2ab$. The area of the quadrilateral in the center is $(a^2 + 2ab + b^2) - 2ab$ or $a^2 + b^2$.

If the quadrilateral in the center is a square then its area also equals $c^2$. You now need to show that it is a square. You know that all the sides have length $c$, but you also need to show that the angles are right angles. The two acute angles in the right triangle, along with any angle of the quadrilateral, add up to 180°. The acute angles in a right triangle add up to 90°. Therefore the quadrilateral angle measures 90° and the quadrilateral is a square. If it is a square with side length $c$, then its area is $c^2$. So, $a^2 + b^2 = c^2$, which proves the Pythagorean Theorem.

NCTM STANDARDS

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LESSON OBJECTIVES

- Understand the Pythagorean Theorem more deeply
- Practice using geometry tools
- Learn new vocabulary

SHARING IDEAS

You might make a transparency of the Dissection of Squares worksheets for students to use in presenting their ideas.

Ask about symmetry in the dissected square on the hypotenuse. The method of the Investigation gives 4-fold rotational symmetry.

[Link] The Pythagorean Theorem is a special case of the Law of Cosines $c^2 = a^2 + b^2 - 2ab \cos C$, where $\angle C$ is the angle opposite side $c$ when $m\angle C = 90^\circ$, we have $\cos C = 0$.

[Ask] “What is the longest side of a right triangle?” “Is it the same as the longest leg?” [The hypotenuse, not the longest leg, is the longest side.] If you ask why the longest side is always the hypotenuse and what can be said about the longer of the two legs, you can review the Triangle Inequality Conjecture and the Side-Angle Inequality Conjecture.

[Ask] “What is a theorem?” [It’s a conjecture that has been proved deductively within a deductive system.] So far in this course no axiom system has been developed, so there are no real theorems; this conjecture is called a theorem in this book because it has been proved within an axiom system and it’s so well known by that name.

If students are not very familiar with the Pythagorean Theorem, ask how this theorem about areas of squares might be used to calculate lengths. Direct students’ attention to the two examples.
The Pythagorean Theorem works for right triangles, but does it work for all triangles? A quick check demonstrates that it doesn't hold for other triangles.

For an interactive version of this sketch, visit [www.keymath.com/DG](http://www.keymath.com/DG).

Let's look at a few examples to see how you can use the Pythagorean Theorem to find the distance between two points.

### EXAMPLE A

How high up on the wall will a 20-foot ladder touch if the foot of the ladder is placed 5 feet from the wall?

**Solution**

The ladder is the hypotenuse of a right triangle, so

\[ a^2 + b^2 = c^2. \]

\[ (5)^2 + (h)^2 = (20)^2 \]

\[ 25 + h^2 = 400 \]

\[ h^2 = 375 \]

\[ h = \sqrt{375} \approx 19.4 \]

The top of the ladder will touch the wall about 19.4 feet up from the ground.

Notice that the exact answer in Example A is \( \sqrt{375} \). However, this is a practical application, so you need to calculate the approximate answer.

### EXAMPLE B

Find the area of the rectangular rug if the width is 12 feet and the diagonal measures 20 feet.

**Solution**

Use the Pythagorean Theorem to find the length.

\[ a^2 + b^2 = c^2 \]

\[ (12)^2 + (L)^2 = (20)^2 \]

\[ 144 + L^2 = 400 \]

\[ L^2 = 256 \]

\[ L = \sqrt{256} \]

\[ L = 16 \]

The length is 16 feet. The area of the rectangle is \( 12 \times 16 \), or 192 square feet.

**Closing the Lesson**

The Pythagorean Theorem is a claim about areas of squares built on the sides of a right triangle: The area of the square on the hypotenuse is the sum of the areas of the squares on the legs. Its primary applications are in finding the length of one side of a right triangle in which the lengths of the other two sides are known. The theorem can be proved by dissection.
In Exercises 1–11, find each missing length. All measurements are in centimeters. Give approximate answers accurate to the nearest tenth of a centimeter.

1. \(a = \sqrt{12}\) cm
2. \(c = \sqrt{19.2}\) cm
3. \(a = \sqrt{5.3}\) cm
4. \(d = \sqrt{10}\) cm
5. \(s = \sqrt{26}\) cm
6. \(c = \sqrt{8.5}\) cm
7. \(b = \sqrt{24}\) cm
8. \(x = \sqrt{3.6}\) cm
9. The base is a circle. \(x = \sqrt{40}\) cm
10. \(s = \sqrt{3.5}\) cm
11. \(r = \sqrt{13}\) cm

12. A baseball infield is a square, each side measuring 90 feet. To the nearest foot, what is the distance from home plate to second base? 127 ft
13. The diagonal of a square measures 32 meters. What is the area of the square? \(512\) m²
14. What is the length of the diagonal of a square whose area is 64 cm²?
15. The lengths of the three sides of a right triangle are consecutive integers. Find them. \(3, 4, 5\)
16. A rectangular garden 6 meters wide has a diagonal measuring 10 meters. Find the perimeter of the garden. 28 m
17. The area of the large square is \(4 \cdot \text{area of triangle} + \text{area of small square.}\)

\[
c^2 = 4 \cdot \frac{1}{2} \cdot ab + (b - a)^2 \\
c^2 = 2ab + b^2 - 2ab + a^2 \\
c^2 = a^2 + b^2
\]

Exercise 18  This problem illustrates a special case in which SSA is a congruence shortcut. That is, when the non-included angle is a right angle, there is only one triangle that can be formed by SSA. This shortcut is sometimes called HL (Hypotenuse-Leg). Students can prove HL using the thinking process used in the solution to this exercise. Given two right triangles with congruent, corresponding hypotenuses with length \(c\) and corresponding legs with length \(a\), use the Pythagorean Theorem to show that the other two corresponding legs are congruent; the triangles are congruent by SSS. Students will prove the HL Theorem in Lesson 13.7

21. Mark the unnamed angles as shown in the figure below.

By the Linear Pair Conjecture, \(p + 120^\circ = 180^\circ \therefore p = 60^\circ\). By AIA, \(m = q\). By the Triangle Sum Conjecture, \(q + p + n = 180^\circ\). Substitute \(m = q\) and \(p = 60^\circ\) to get \(m + 60^\circ + n = 180^\circ\). \(\therefore m + n = 120^\circ\).

Or use the Exterior Angle Conjecture \(q + n = 120^\circ\). By AIA \(q = m\). Substituting, \(m + n = 120^\circ\).
CREATING A GEOMETRY FLIP BOOK

Have you ever fanned the pages of a flip book and watched the pictures seem to move? Each page shows a picture slightly different from the previous one. Flip books are basic to animation technique. For more information about flip books, see www.keymath.com/DG.

Here are two dissections that you can animate to demonstrate the Pythagorean Theorem. (You used another dissection in the Investigation The Three Sides of a Right Triangle.)

You could also animate these drawings to demonstrate area formulas.

Choose one of the animations mentioned above and create a flip book that demonstrates it. Be ready to explain how your flip book demonstrates the formula you chose.

Here are some practical tips.
- Draw your figures in the same position on each page so they don’t jump around when the pages are flipped. Use graph paper or tracing paper to help.
- The smaller the change from picture to picture, and the more pictures there are, the smoother the motion will be.
- Label each picture so that it’s clear how the process works.

OUTCOMES
- Movement in the flip book is smooth.
- The student can explain the dissection used.
- The student adds other animation, for example, drawing a hand flipping the pages of a flip book, thus creating a flip book of a flip book!
- An animation is created using geometry software.

Supporting the project

Suggest that students use a small graph paper tablet to help keep the nonmoving features in the same position from page to page. They can glue tracing papers of the flip book onto cards for a firmer flip.

[Context] Eadweard Muybridge, the man who created The Horse in Motion, was born in England but moved to the United States as a boy. As part of scientific research to improve techniques of horseracing, he devised an ingenious method of setting up a dozen or more cameras to go off in rapid sequence. He was a leader in the early days of photography, improving on existing methods, including the art of film developing. He invented a precursor to the movie projector based on the idea of a zoetrope, a slitted drum with pictures on the inside that is spun to show the illusion of motion when the pictures are watched through the slits.

EXTENSIONS
A. Have students research and try one of numerous other dissections that demonstrate the Pythagorean Theorem.
B. Use Take Another Look activity 1, 2, or 3 on pages 501–502.
In Lesson 9.1, you saw that if a triangle is a right triangle, then the square of the length of its hypotenuse is equal to the sum of the squares of the lengths of the two legs. What about the converse? If $x$, $y$, and $z$ are the lengths of the three sides of a triangle and they satisfy the Pythagorean equation, $a^2 + b^2 = c^2$, must the triangle be a right triangle? Let's find out.

**Investigation**

Is the Converse True?

Three positive integers that work in the Pythagorean equation are called Pythagorean triples. For example, $8-15-17$ is a Pythagorean triple because $8^2 + 15^2 = 17^2$. Here are nine sets of Pythagorean triples.

- $3-4-5$
- $5-12-13$
- $7-24-25$
- $8-15-17$
- $6-8-10$
- $10-24-26$
- $16-30-34$
- $9-12-15$
- $12-16-20$

**Step 1** Select one set of Pythagorean triples from the list above. Mark off four points, $A$, $B$, $C$, and $D$, on a string to create three consecutive lengths from your set of triples.

**Step 2** Loop three paper clips onto the string. Tie the ends together so that points $A$ and $D$ meet.

**Step 3** Three group members should each pull a paper clip at point $A$, $B$, or $C$ to stretch the string tight.

You might let the class do this investigation outside, with the students themselves acting as “rope stretchers” of long ropes.

**Guiding the Investigation**

Pose this problem: “What can you say about triangles in which the side lengths satisfy the equation $a^2 + b^2 = c^2$?” As you circulate, encourage groups to try various triples from the list in the student book or to make up their own lengths, perhaps not integers.

As needed, remind students that the converse of a true statement might be false.

The Pythagorean triples are arranged in families. [Ask] “How are the triples in a column related?” [They are multiples of the first triple in that column.] “What would be the next triple in the second column?” [15, 36, 39] “Why is 3-4-5 or 5-12-13 called a primitive Pythagorean triple?” [The numbers in the triple have no common factor.]

**LESSON OBJECTIVES**

- Discover the Converse of the Pythagorean Theorem
- Learn new vocabulary
- Develop reading comprehension, problem-solving skills, and cooperative behavior
**Step 4** Students can also use a corner of a piece of paper (or of some other object such as a carpenter’s square) to verify a right angle.

**SHARING IDEAS**

Have students show their work with a variety of lengths. Let the class discuss measurement errors. Elicit the idea that a slight error in the measurement of a length can result in a measurably different angle. Keep asking students whether they think the converse is always true. Agree on a statement of the conjecture without resolving the question of truth.

*Ask* “Will the conjecture hold if you use different measurement units so that the triples change, possibly to non-integers?”

*Ask* “Suppose one triangle has sides whose lengths are a Pythagorean triple and another triangle has sides whose lengths are a multiple of that Pythagorean triple. How are the triangles related?” [They are similar.] Students can try drawing such triangles and looking for patterns, but you need not answer the question yet. It foreshadows the ideas of similarity in Chapter 11.

Also ask whether students who believe the conjecture is true can prove it deductively. Then have the class read the proof outline in the student book.

**MAKING THE CONNECTION**

Pythagoras himself is believed to have studied in Egypt, and he may have learned the triangle relationship there. Although some historians discount the rope stretchers tale, there’s no doubt that Egyptian mathematicians knew the relationship.
Proof
Ask students to critique this outline in order to deepen their understanding of it. As they discuss it, monitor their facial expressions and try to include students who seem to have ideas but aren’t speaking up. You may want to let the discussion lead to filling in details as a class and then writing up a good model proof. Or, if your students are fairly comfortable with proof, you can challenge them to write up the details as homework.

Assessing Progress
You can check how well students can measure lengths and angles, experiment systematically, and follow a deductive proof.

Closing the Lesson
The Converse of the Pythagorean Theorem is true and can be used to determine right angles. A common proof actually uses the Pythagorean Theorem itself.

BUILDING UNDERSTANDING
These exercises help students practice both the Pythagorean Theorem and its converse.

ASSIGNING HOMEWORK

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Helping with the Exercises

Exercise 2 This is a good place to point out the power of recognizing Pythagorean multiples. The triple 50-120-130 is based on the Pythagorean primitive 5-12-13, so sides of those lengths form a right triangle.

Exercise 7 Students’ justifications might cite Pythagorean multiples. The numbers 9, 12, and 18 have a common factor of 3, so a primitive would be 3-4-6. This is not a Pythagorean triple; the familiar 3-4-5 triple says that if the two legs have lengths 3 and 4, the hypotenuse must have length 5 to give a right triangle.

Exercise 8 After students decide that the angle isn’t right, you might ask whether students could have known the window frame was rectangular if the angle had turned out to be right. Having one right angle is not a sufficient condition for a quadrilateral to be a rectangle.

Proof: Converse of the Pythagorean Theorem

Conjecture: If the lengths of the three sides of a triangle work in the Pythagorean equation, then the triangle is a right triangle.

Given: \( a, b, c \) are the lengths of the sides of \( \triangle ABC \) and \( a^2 + b^2 = c^2 \)

Show: \( \triangle ABC \) is a right triangle

Plan: Begin by constructing a second triangle, right triangle \( \triangle DEF \) (with \( \angle F \) a right angle), with legs of lengths \( a \) and \( b \) and hypotenuse of length \( x \). The plan is to show that \( x = c \), so that the triangles are congruent. Then show that \( \angle C \) and \( \angle F \) are congruent. Once you show that \( \angle C \) is a right angle, then \( \triangle ABC \) is a right triangle and the proof is complete.
In Exercises 9–11, find \( y \).

9. Both quadrilaterals are squares. \( y = 25 \text{ cm} \)

![Square diagram]

10. \( \triangle \)

![Triangle diagram]

11. \( y = 17.3 \text{ m} \)

12. The lengths of the three sides of a right triangle are consecutive even integers. Find them. \( 6, 8, 10 \)

13. Find the area of a right triangle with hypotenuse length 17 cm and one leg length 15 cm. \( 60 \text{ cm}^2 \)

![Right triangle diagram]

14. How high on a building will a 15-foot ladder touch if the foot of the ladder is 5 feet from the building? \( 14.1 \text{ ft} \)

15. The congruent sides of an isosceles triangle measure 6 cm, and the base measures 8 cm. Find the area. \( 17.9 \text{ cm}^2 \)

Review

3.8 21. Identify the point of concurrency from the construction marks. centroid

![Concurrency diagram]

18. Because \( \triangle DEF \) is a right triangle, \( a^2 + b^2 = x^2 \). By substitution, \( c^2 = x^2 \) and \( c = x \). Therefore, \( \triangle EFD \cong \triangle BCA \) by SSS and \( \angle C \cong \angle F \) by CPCTC. Hence, \( \angle C \) is a right angle and \( \triangle BCA \) is a right triangle.

Exercise 9 Here is another situation in which Pythagorean triples can be applied. For the triangle with \( y \) as the hypotenuse, the legs have lengths 15 cm and 20 cm. Each length has a common factor of 5 cm. Because 3–4–5 is a common Pythagorean primitive, the hypotenuse must have length 5(5 cm) = 25 cm.

Exercise 10 This exercise provides groundwork for Chapter 12 work with the unit circle in trigonometry.

Exercises 12–16 Encourage students to draw pictures.

Exercise 12 Students might miss the condition that the integers are even. If appropriate, [Ask] “If \( x \) is the first even integer, how would the second integer be described?” [\( x + 2 \)] “Is there only one triple of consecutive even integers?” [Half of any such triple is a triple of consecutive integers, say, \( x = 1, x, x + 1 \). The algebraic equation \( (x - 1)^2 + x^2 = (x + 1)^2 \) has only two solutions: \( x = 0 \) and \( x = 4 \).]

Exercise 15 [Ask] “In an isosceles triangle, what does an altitude from the vertex angle to the base do to the base?” [It is the perpendicular bisector of the base; it is the same as the median.]

Exercise 16 [Language] Linear feet refers to the length of the fence, whereas square feet measures area.

Exercise 17 [Alert] Some students will have difficulty visualizing the box. They might create a three-dimensional model. Students are being gradually introduced to the Pythagorean Theorem in three dimensions. As in Exercises 11 and 13–16, exactness of answers will vary. The answers given assume that measurements given in the exercises are exact and thus answers to the nearest square cm or nearest tenth of a cm are reasonable. In real life the exactness of an answer depends on how it will be used and further knowledge of given measurements.
**Exercise 22** This exercise can be approached through finding that the complement of \(x\) is \(90 - \frac{\pi}{2}\), through using the properties of an isosceles right triangle, or by considering the limit as a secant line through \(C\) rotates to become the tangent line (and the intercepted arc becomes the arc with measure \(a\)).

**Exercise 23** If students are having difficulty, ask how they might count triangles systematically. One approach is to consider where the vertex angle might go. Symmetry helps.

**Exercise 24** As needed, focus students’ attention on the number of square faces being added at each step.

**EXTENSIONS**

**A.** Pose this problem: If the sum of the squares of the lengths of the two shorter sides of a triangle is less than the square of the length of the longest side, what can you conjecture about the angle opposite the longest side? If the sum of the squares of the lengths of the two shorter sides of a triangle is greater than the square of the length of the longest side, what can you conjecture about the angle opposite the longest side? [If the sum is less, the angle is obtuse. If the sum is greater, the angle is acute.]

**B.** Have students use geometry software to draw triangles. Have them label and measure angles and sides, calculate squares of the lengths of the sides or construct squares on the sides and measure their area, and drag vertices until the sum of the squares of the lengths of the two smallest sides equals the square of the length of the largest side. Students should find a right angle.

---

**6.3 22.** Line \(CF\) is tangent to circle \(D\) at \(C\). The arc measure of \(CE\) is \(a\). Explain why \(x = \left(\frac{1}{2}\right)a\).

**23.** What is the probability of randomly selecting three points that form an isosceles triangle from the 10 points in this isometric grid? [3/10]

**24.** If the pattern of blocks continues, what will be the surface area of the 50th solid in the pattern? [790 square units]

**25.** Sketch the solid shown, but with the two blue cubes removed and the red cube moved to cover the visible face of the green cube.

---

**IMPROVING YOUR ALGEBRA SKILLS**

**Algebraic Sequences III**

Find the next three terms of this algebraic sequence.

\[ x^1, 9x^2y, 36x^3y^2, 84x^4y^3, 126x^5y^4, 168x^6y^5, \ldots, \ldots, \ldots \]

---

**IMPROVING ALGEBRA SKILLS**

Students might think of number combinations, Pascal’s triangle, or symmetry. Or they might see this pattern: After the first term, each coefficient can be determined by multiplying the previous coefficient by the exponent on \(x\) in the previous term and then dividing by the number of the term (counting from zero). For example, the term that includes \(x^6\) is the third term, so its coefficient is \(\frac{7!}{3!} = 84\). The next three terms in the sequence are \(36x^2y^7\), \(9xy^8\), and \(y^9\).
Radical Expressions

When you work with the Pythagorean Theorem, you often get radical expressions, such as $\sqrt{50}$. Until now you may have left these expressions as radicals, or you may have found a decimal approximation using a calculator. Some radical expressions can be simplified. To simplify a square root means to take the square root of any perfect-square factors of the number under the radical sign. Let’s look at an example.

**EXAMPLE A**

Simplify $\sqrt{50}$.

**Solution**

One way to simplify a square root is to look for perfect-square factors.

$$\sqrt{50} = \sqrt{25 \cdot 2} = \sqrt{25} \cdot \sqrt{2} = 5\sqrt{2}$$

25 is a perfect square, so you can take its square root.

Another approach is to factor the number as far as possible with prime factors.

$$\sqrt{50} = \sqrt{3 \cdot 5 \cdot 2} = \sqrt{5^2 \cdot 2} = 5\sqrt{2}$$

Squaring and taking the square root are inverse operations—they undo each other. So, $\sqrt{50}$ equals 5.

You might argue that $5\sqrt{2}$ doesn’t look any simpler than $\sqrt{50}$. However, in the days before calculators with square root buttons, mathematicians used paper-and-pencil algorithms to find approximate values of square roots. Working with the smallest possible number under the radical made the algorithms easier to use.

Giving an exact answer to a problem involving a square root is important in a number of situations. Some patterns are easier to discover with simplified square roots than with decimal approximations. Standardized tests often express answers in simplified form. And when you multiply radical expressions, you often have to simplify the answer.

**LESSON OBJECTIVES**

- Learn to simplify square roots
- Learn to multiply radical expressions

**NCTM STANDARDS**

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EXAMPLE B
You may want to mention that the commutative property of multiplication allows you to rewrite $3 \cdot \sqrt{6} \cdot 5 \cdot \sqrt{2}$ as $3 \cdot 5 \cdot \sqrt{6} \cdot \sqrt{2}$.

Solution
To multiply radical expressions, associate and multiply the quantities outside the radical sign, and associate and multiply the quantities inside the radical sign.

$(3\sqrt{6})(5\sqrt{2}) = 3 \cdot 5 \cdot \sqrt{6} \cdot \sqrt{2} = 15 \cdot \sqrt{12} = 15 \cdot \sqrt{4} \cdot \sqrt{3} = 15 \cdot 2\sqrt{3} = 30\sqrt{3}$

Exercises
In Exercises 1–5, express each product in its simplest form.

1. $(\sqrt{2})(\sqrt{2}) \sqrt{6}$  
2. $(\sqrt{5})^5$  
3. $(3\sqrt{6})(2\sqrt{3})$  
4. $(7\sqrt{3})^2$  
5. $(2\sqrt{2})^2$  

In Exercises 6–20, express each square root in its simplest form.

6. $\sqrt{18}$  
7. $\sqrt{40}$  
8. $\sqrt{75}$  
9. $\sqrt{85}$  
10. $\sqrt{96}$  
11. $\sqrt{576}$  
12. $\sqrt{720}$  
13. $\sqrt{722}$  
14. $\sqrt{784}$  
15. $\sqrt{828}$  
16. $\sqrt{2952}$  
17. $\sqrt{5248}$  
18. $\sqrt{8200}$  
19. $\sqrt{11808}$  
20. $\sqrt{16072}$  
21. What is the next term in the pattern? $\sqrt{2952}, \sqrt{5248}, \sqrt{8200}, \sqrt{11808}, \sqrt{16072}, \ldots$

IMPROVING YOUR VISUAL THINKING SKILLS
Folding Cubes II
Each cube has designs on three faces. When unfolded, which figure at right could it become?

1. A. B. C. D.
2. A. B. C. D.

Helping with the Exercises
Exercise 2 Squaring and taking the square root are opposite operations.
Exercise 9 It is possible to factor 85 as $5 \cdot 17$, but neither factor is a perfect square.
Exercise 21 If students are mystified, refer them to Exercises 16–20.

BUILDING UNDERSTANDING
Though students are taught to simplify radical expressions, they are not taught to rationalize the denominators. You might decide to teach that also.

ASSIGNING HOMEWORK
Essential 1–21 odds
Group 2–20 evens
Two Special Right Triangles

In this lesson you will use the Pythagorean Theorem to discover some relationships between the sides of two special right triangles.

One of these special triangles is an isosceles right triangle, also called a 45°-45°-90° triangle. Each isosceles right triangle is half a square, so they show up often in mathematics and engineering. In the next investigation, you will look for a shortcut for finding the length of an unknown side in a 45°-45°-90° triangle.

Investigation 1

Isosceles Right Triangles

Step 1 Sketch an isosceles right triangle. Label the legs $l$ and the hypotenuse $h$.

Step 2 Pick any integer for $l$, the length of the legs. Use the Pythagorean Theorem to find $h$. Simplify the square root.

Step 3 Repeat Step 2 with several different values for $l$. Share results with your group. Do you see any pattern in the relationship between $l$ and $h$?

Step 4 State your next conjecture in terms of length $l$.

Isosceles Right Triangle Conjecture

In an isosceles right triangle, if the legs have length $l$, then the hypotenuse has length $\frac{\sqrt{2}}{2} l$.

Guiding Investigation 1

Step 3 If students are having difficulty seeing a pattern, suggest that they systematically try consecutive integers and make a table of the results.
**Guiding Investigation 2**

Steps 1–3 Students can use patty paper to answer some of the questions in Steps 2 and 3 without doing the measurements in Step 1. They might also recall that every altitude is a median.

**SHARING IDEAS**

After the class reaches consensus about what conjectures to record in their notebooks, ask how to restate the conjectures using ratios. [For a 45°-45°-90° triangle, the side lengths have the ratio 1:1:√2; for a 30°-60°-90° triangle, they have the ratio 1:√3:2.]

Ask whether these ratios can be represented geometrically, remembering that square roots are often shown as sides of squares. Students can draw the triangles on dot paper (or isometric dot paper) and calculate the areas of appropriate squares.

**[Ask]** “How do you know which angle is the 30° angle in a 30°-60°-90° triangle?” [By the Side-Angle Inequality Conjecture, it’s the angle opposite the shortest side.]

**[Ask]** “How can these special triangles be constructed with straightedge and compass?” [Students can construct perpendicular lines and then lay out equal segments from the intersection point to get a 45°-45°-90° triangle. Now that they know the 30°-60°-90° Triangle Conjecture, they can construct one leg and a hypotenuse to determine the third side.]

Wonder aloud whether the 30°-60°-90° Triangle Conjecture can be proved for all triangles, even if none of the lengths are integers. After students have made suggestions, direct their attention to the proof in the student book, asking them to critique and rewrite the reasoning in order to understand it better.

**Investigation 2**

**30°-60°-90° Triangles**

Let’s start by using a little deductive thinking to find the relationships in 30°-60°-90° triangles. Triangle ABC is equilateral, and CD is an altitude.

What are m∠A and m∠B? What are m∠ACD and m∠BCD? What are m∠ADC and m∠BDC?

Is ΔADC ≅ ΔBCD? Why?

Is AD = BD? Why? How do AC and AD compare?

In a 30°-60°-90° triangle, will this relationship between the hypotenuse and the shorter leg always hold true? Explain.

Sketch a 30°-60°-90° triangle. Choose any integer for the length of the shorter leg. Use the relationship from Step 3 and the Pythagorean Theorem to find the length of the other leg. Simplify the square root.

Repeat Step 4 with several different values for the length of the shorter leg. Share results with your group. What is the relationship between the lengths of the two legs? You should notice a pattern in your answers.

State your next conjecture in terms of the length of the shorter leg, a.

**Assessing Progress**

Assess students’ ability to generate isosceles right triangles and equilateral triangles, to simplify square roots, to measure angles, to find patterns, and to follow a deductive proof. Also check their understanding of altitudes of isosceles triangles, SAS, and the Pythagorean Theorem.

**Closing the Lesson**

In a 45°-45°-90° triangle, if the legs have length l, then the hypotenuse has length l√2. In a 30°-60°-90° triangle, if the leg opposite the 30° angle has length a, then the hypotenuse has length 2a and the other leg has length a√3.
You can use algebra to verify that the conjecture will hold true for any 30°-60°-90° triangle.

**Proof: 30°-60°-90° Triangle Conjecture**

\[
\begin{align*}
(2a)^2 &= a^2 + b^2 & \text{Start with the Pythagorean Theorem.} \\
4a^2 &= a^2 + b^2 & \text{Square } 2a. \\
3a^2 &= b^2 & \text{Subtract } a^2 \text{ from both sides.} \\
a\sqrt{3} &= b & \text{Take the square root of both sides.}
\end{align*}
\]

Although you investigated only integer values, the proof shows that any number, even a non-integer, can be used for \(a\). You can also demonstrate this property for integer values on isometric dot paper.

**EXERCISES**

In Exercises 1–8, use your new conjectures to find the unknown lengths. All measurements are in centimeters.

1. \(a = \sqrt[2]{72} \text{ cm}\)

2. \(b = \frac{\sqrt[10]{3}}{13} \text{ cm}\)

3. \(a = \frac{\sqrt[10]{3}}{10} \text{ cm}, b = \frac{\sqrt[10]{3}}{5} \text{ cm}\)

4. \(c = \sqrt[10]{3} \text{ cm}, d = \frac{\sqrt[10]{3}}{10} \text{ cm}\)

5. \(e = \frac{\sqrt[10]{3}}{34} \text{ cm}, f = \frac{\sqrt[10]{3}}{17} \text{ cm}\)

6. What is the perimeter of square SQRE? 72 cm

**BUILDING UNDERSTANDING**

The exercises give practice with the two special right triangles.

**ASSIGNING HOMEWORK**

- **Essential** 1–11
- **Performance assessment** 18
- **Portfolio** 17
- **Group** 12–16
- **Review** 19–23

**Helping with the Exercises**

Exercise 3 If students aren’t sure what to do, suggest that in a 30°-60°-90° triangle it often helps to locate the shortest side first. 
[Ask] “Where is the 30-degree angle?”
CHAPTER 9 The Pythagorean Theorem

Exercise 7  This is another example of using the Pythagorean Theorem in three dimensions. [Ask] “How would you generalize the Pythagorean Theorem to three dimensions?” In a right rectangular prism, the space diagonal (d) can be found from the three dimensions of the prism (a, b, c): \( d^2 = a^2 + b^2 + c^2 \).

Exercise 10  In situations in which you want to find the coordinates of a point, it’s often useful to draw segments whose lengths are those coordinates. [Link] Students will work with the unit circle in trigonometry.

12. possible answer:

13. possible answer:

Exercise 16  Ask students how to find a special triangle in a picture of this situation.

Exercise 18  Students using geometry software might discover that letting the hypotenuses (not the right angles) coincide produces a slightly larger triangle.

15. \( c^2 = x^2 + x^2 \)  Start with the Pythagorean Theorem.

\( c^2 = 2x^2 \)  Combine like terms.

\( c = x\sqrt{2} \)  Take the square root of both sides.
The coach. The team has enough money for require 23 round trips for the 11 players and bring the boat back on the next trip. This will must cross and leave one on the other side to each Monster crosses, the two Smallville players makes no sense for one to return, so before
IMPROVING YOUR VISUAL THINKING

Mudville Monsters

The 11 starting members of the Mudville Monsters football team and their coach, Osgood Gipper, have been invited to compete in the Smallville Punt, Pass, and Kick Competition. To get there, they must cross the deep Smallville River. The only way across is with a small boat owned by two very small Smallville football players. The boat holds just one Monster visitor or the two Smallville players. The Smallville players agree to help the Mudville players across if the visitors agree to pay $5 each time the boat crosses the river. How long is the rope?

Exercise 20 This mini-investigation foreshadows area ratios in Lesson 11.5.

19. Construct an isosceles right triangle with legs of length \( a \), construct a 30°-60°-90° triangle with legs of lengths \( a \) and \( a\sqrt{3} \), and construct a right triangle with legs of lengths \( a\sqrt{2} \) and \( a\sqrt{3} \).

20. Areas: 4.5 \( \pi \) cm\(^2\), 8 \( \pi \) cm\(^2\), 12.5 \( \pi \) cm\(^2\), 4.5 \( \pi \) + 8 \( \pi \) = 12.5 \( \pi \), that is, the sum of the areas of the semicircles on the two legs is equal to the area of the semicircle on the hypotenuse.

22. Extend the rays that form the right angle. \( m\angle 4 + m\angle 5 = 180° \) by the Linear Pair Conjecture, and it’s given that \( m\angle 5 = 90° \). \( m\angle 2 + m\angle 3 + m\angle 4 = m\angle 2 + m\angle 3 + 90° = 180° \). \( m\angle 2 + m\angle 3 = 90° \). \( m\angle 3 = m\angle 1 \) by AIA. \( m\angle 1 + m\angle 2 = 90° \).

EXTENSION
Use Take Another Look activity 4 or 5 on page 502.

23. The lateral surface area of the cone below is unwrapped into a sector. What is the angle at the vertex of the sector? 80°

3. Given the segment with length \( a \) below, construct segments with lengths \( a\sqrt{2} \), \( a\sqrt{3} \), and \( a\sqrt{5} \).

20. Mini-Investigation Draw a right triangle with sides of lengths 6 cm, 8 cm, and 10 cm. Locate the midpoint of each side. Construct a semicircle on each side with the midpoints of the sides as centers. Find the area of each semicircle. What relationship do you notice among the three areas?

9.1 21. The jiuzhang suanshu is an ancient Chinese mathematics text of 246 problems. Some solutions use the gou gu, the Chinese name for what we call the Pythagorean Theorem. The gou gu reads \( (gou)^2 + (gu)^2 = (xian)^2 \). Here is a gou gu problem translated from the ninth chapter of jiuzhang.

A rope hangs from the top of a pole with three chih of it lying on the ground. When it is tightly stretched so that its end just touches the ground, it is eight chih from the base of the pole. How long is the rope? \( \frac{12}{3} = 4.3 \) chih

22. Explain why \( m\angle 1 + m\angle 2 = 90° \).

23. The lateral surface area of the cone below is unwrapped into a sector. What is the angle at the vertex of the sector? 80°

19. Construct an isosceles right triangle with legs of length \( a \), construct a 30°-60°-90° triangle with legs of lengths \( a \) and \( a\sqrt{3} \), and construct a right triangle with legs of lengths \( a\sqrt{2} \) and \( a\sqrt{3} \).

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EXTENSION
Use Take Another Look activity 4 or 5 on page 502.

19. Construct an isosceles right triangle with legs of length \( a \), construct a 30°-60°-90° triangle with legs of lengths \( a \) and \( a\sqrt{3} \), and construct a right triangle with legs of lengths \( a\sqrt{2} \) and \( a\sqrt{3} \).
A Pythagorean Fractal

If you wanted to draw a picture to state the Pythagorean Theorem without words, you'd probably draw a right triangle with squares on each of the three sides. This is the way you first explored the Pythagorean Theorem in Lesson 9.1.

Another picture of the theorem is even simpler: a right triangle divided into two right triangles. Here, a right triangle with hypotenuse $c$ is divided into two smaller triangles, the smaller with hypotenuse $a$ and the larger with hypotenuse $b$. Clearly, their areas add up to the area of the whole triangle. What's surprising is that all three triangles have the same angle measures. Why? Though different in size, the three triangles all have the same shape. Figures that have the same shape but not necessarily the same size are called similar figures. You'll use these similar triangles to prove the Pythagorean Theorem in a later chapter.

A beautifully complex fractal combines both of these pictorial representations of the Pythagorean Theorem. The fractal starts with a right triangle with squares on each side. Then similar triangles are built onto the squares. Then squares are built onto the new triangles, and so on. In this exploration, you'll create this fractal.

**LESSON OBJECTIVES**

- Explore a right triangle fractal
- Look for and describe patterns in the fractal

**NCTM STANDARDS**

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Activity

The Right Triangle Fractal

You will need
- the worksheet
  The Right Triangle Fractal (optional)

Step 1
Use The Geometer’s Sketchpad to create the fractal on page 480. Follow the Procedure Note.

Notice that each square has two congruent triangles on two opposite sides. Use a reflection to guarantee that the triangles are congruent.

After you successfully make the Pythagorean fractal, you’re ready to investigate its fascinating patterns.

Step 2
First, try dragging a vertex of the original triangle.

Step 3
Does the Pythagorean Theorem still apply to the branches of this figure? That is, does the sum of the areas of the branches on the legs equal the area of the branch on the hypotenuse? See if you can answer without actually measuring all the areas. yes

Consider your original sketch to be a single right triangle with a square built on each side. Call this sketch Stage 0 of your fractal. Explore these questions.

a. At Stage 1, you add three triangles and six squares to your construction. On a piece of paper, draw a rough sketch of Stage 1. How much area do you add to this fractal between Stage 0 and Stage 1? (Don’t measure any areas to answer this.)

b. Draw a rough sketch of Stage 2. How much area do you add between Stage 1 and Stage 2?

c. How much area is added at any new stage?

d. A true fractal exists only after an infinite number of stages. If you could build a true fractal based on the construction in this activity, what would be its total area?

Give the same color and shade to sets of squares that are congruent. What do you notice about these sets of squares other than their equal area? Describe any patterns you find in sets of congruent squares.

Describe any other patterns you can find in the Pythagorean fractal.

Step 4
At each stage, the sum of the areas of the squares added to each branch equals the area of the original square on that branch.

Step 4a
At Stage 1, the area of the triangle on the hypotenuse branch equals the area of the original triangle, and the sum of the areas of the triangles on the other two branches also equals the area of the original triangle. The sum of the two added squares equals the area of one of the original squares by the Pythagorean Theorem. Thus the total added area is twice the area of the original triangle plus the areas of all the original squares, or the area of the Stage 0 figure plus the area of the original triangle.

Step 4b
The total added area is, again, the area of the Stage 0 figure plus the area of the original triangle.

Step 4c
At every new stage, the area added equals the area of the Stage 0 figure plus the area of the original triangle.

Step 4d
Its area would be infinite.

Step 5
Sample observations: Each square on the hypotenuse branch has a mirror image on one of the leg branches. The squares of equal area on the leg branches all face the same direction; that is, corresponding sides in all the squares are parallel. The number of squares of a particular area on the leg branches is one more than the number of squares of the next largest area on those branches.

Step 6
Students’ responses will vary. Check them for validity and justification.

Step 5
Students can see many patterns. For example, there is reflectional symmetry over a midsegment of the square on the original hypotenuse.

Step 6
Students’ responses will vary. Check them for validity and justification.
Story Problems

You have learned that drawing a diagram will help you to solve difficult problems. By now you know to look for many special relationships in your diagrams, such as congruent polygons, parallel lines, and right triangles.

What is the longest stick that will fit inside a 24-by-30-by-18-inch box?

Draw a diagram.

You can lay a stick with length \( d \) diagonally at the bottom of the box. But you can position an even longer stick with length \( x \) along the diagonal of the box, as shown. How long is this stick?

Both \( d \) and \( x \) are the hypotenuses of right triangles, but finding \( d^2 \) will help you find \( x \).

\[
\begin{align*}
30^2 + 24^2 &= d^2 \\
900 + 576 &= d^2 \\
1476 &= d^2 \\
\sqrt{1476} &= d \\
38.4 &= d
\end{align*}
\]

\[
\begin{align*}
24^2 + 18^2 &= x^2 \\
576 + 324 &= x^2 \\
900 &= x^2 \\
\sqrt{900} &= x \\
30 &= x
\end{align*}
\]

The longest possible stick is about 38.4 in.

**EXAMPLE**

**Solution**

The distance \( x \) is called the space diagonal. To find its length, you need \( d^2 \), but you don’t need to find \( d \).

Assessing Progress

Through their work on the exercises, you can assess students’ understanding of the Pythagorean Theorem.

**SHARING IDEAS**

Have several groups prepare some of their solutions to the exercises on transparencies and share them with the class. Keep asking whether the results seem reasonable.

**Closing the Lesson**

Emphasize that a good first step in solving application problems is to draw a picture. Then students should examine the picture for common geometric shapes, such as right triangles, and apply what they know about those shapes.

**EXERCISES**

1. A giant California redwood tree 36 meters tall cracked in a violent storm and fell as if hinged. The tip of the once beautiful tree hit the ground 24 meters from the base. Researcher Red Woods wishes to investigate the crack. How many meters up from the base of the tree does he have to climb? 10 m

2. Amir’s sister is away at college, and he wants to mail her a 34 in. baseball bat. The packing service sells only one kind of box, which measures 24 in. by 2 in. by 18 in. Will the box be big enough? No. The space diagonal of the box is 30.1 in.
3. Meteorologist Paul Windward and geologist Rhaina Stone are rushing to a paleontology conference in Pecos Gulch. Paul lifts off in his balloon at noon from Lost Wages, heading east for Pecos Gulch Conference Center. With the wind blowing west to east, he averages a land speed of 30 km/hr. This will allow him to arrive in 4 hours, just as the conference begins. Meanwhile, Rhaina is 160 km north of Lost Wages. At the moment of Paul’s lift off, Rhaina hops into an off-roading vehicle and heads directly for the conference center. At what average speed must she travel to arrive at the same time Paul does?  

4. A 25-foot ladder is placed against a building. The bottom of the ladder is 7 feet from the building. If the top of the ladder slips down 4 feet, how many feet will the bottom slide out? (It is not 4 feet.)

5. The front and back walls of an A-frame cabin are isosceles triangles, each with a base measuring 10 m and legs measuring 13 m. The entire front wall is made of glass 1 cm thick that cost $120/m². What did the glass for the front wall cost?

6. A regular hexagonal prism fits perfectly inside a cylindrical box with diameter 6 cm and height 10 cm. What is the surface area of the prism? What is the surface area of the cylinder?

7. Find the perimeter of an equilateral triangle whose median measures 6 cm.

8. APPLICATION According to the Americans with Disabilities Act, the slope of a wheelchair ramp must be no greater than \( \frac{1}{12} \). What is the length of ramp needed to gain a height of 4 feet? Read the Science Connection on the top of page 484 and then figure out how much force is required to go up the ramp if a person and a wheelchair together weigh 200 pounds.
For Exercises 9 and 10, refer to the above Science Connection about inclined planes.

9. Compare what it would take to lift an object these three different ways.
   a. How much work, in foot-pounds, is necessary to lift 80 pounds straight up 2 feet? 160 ft-lb
   b. If a ramp 4 feet long is used to raise the 80 pounds up 2 feet, how much force, in pounds, will it take? 40 lb
   c. If a ramp 8 feet long is used to raise the 80 pounds up 2 feet, how much force, in pounds, will it take? 20 lb

10. If you can exert only 70 pounds of force and you need to lift a 160-pound steel drum up 2 feet, what is the minimum length of ramp you should set up? 4.6 ft

Review

Making the Connection

Qi qiao is pronounced [chē chēau].

Science CONNECTION

It takes less effort to roll objects up an inclined plane, or ramp, than to lift them straight up. Work is a measure of force applied over distance, and you calculate it as a product of force and distance. For example, a force of 100 pounds is required to hold up a 100-pound object. The work required to lift it 2 feet is 200 foot-pounds. But if you use a 4-foot-long ramp to roll it up, you’ll do the 200 foot-pounds of work over a 4-foot distance. So you need to apply only 50 pounds of force at any given moment.

For Exercises 9 and 10, refer to the above Science Connection about inclined planes.

9. Compare what it would take to lift an object these three different ways.
   a. How much work, in foot-pounds, is necessary to lift 80 pounds straight up 2 feet? 160 ft-lb
   b. If a ramp 4 feet long is used to raise the 80 pounds up 2 feet, how much force, in pounds, will it take? 40 lb
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10. If you can exert only 70 pounds of force and you need to lift a 160-pound steel drum up 2 feet, what is the minimum length of ramp you should set up? 4.6 ft

Recreation CONNECTION

This set of enameled porcelain qi qiao bowls can be arranged to form a 37-by-37 cm square (as shown) or other shapes, or used separately. Each bowl is 10 cm deep. Dishes of this type are usually used to serve candies, nuts, dried fruits, and other snacks on special occasions.

The qi qiao, or tangram puzzle, originated in China and consists of seven pieces—five isosceles right triangles, a square, and a parallelogram. The puzzle involves rearranging the pieces into a square, or hundreds of other shapes (a few are shown below).

Cat  Rabbit  Horse with Rider

9.1 11. If the area of the red square piece is 4 cm², what are the dimensions of the other six pieces?
12. Make a set of your own seven tangram pieces and create the Cat, Rabbit, Swan, and Horse with Rider as shown on page 484.

9.3 13. Find the radius of circle Q. 9.1 14. Find the length of $\overline{AC}$. 6.2 15. The two rays are tangent to the circle. What's wrong with this picture?

12 units

18$\sqrt{2}$ cm

6.2 15. The two rays are tangent to the circle. What's wrong with this picture?

7.1 16. In the figure below, point $A'$ is the image of point A after a reflection over OT. What are the coordinates of $A'$?

$(4, 4\sqrt{3})$

17. Which congruence shortcut can you use to show that $\triangle ABP \cong \triangle DCP$?

SAA

18. Identify the point of concurrency in $\triangle QUO$ from the construction marks.

orthocenter

5.5 19. In parallelogram $QUID$, $m \angle Q = 2x + 5^\circ$ and $m \angle I = 4x - 55^\circ$. What is $m \angle U$?

115°

4.3 20. In $\triangle PRO$, $m \angle P = 70^\circ$ and $m \angle R = 45^\circ$. Which side of the triangle is the shortest?

PO

IMPROVING YOUR VISUAL THINKING SKILLS

Fold, Punch, and Snip

A square sheet of paper is folded vertically, a hole is punched out of the center, and then one of the corners is snipped off. When the paper is unfolded it will look like the figure at right.

Sketch what a square sheet of paper will look like when it is unfolded after the following sequence of folds, punches, and snips.

Fold once.  
Fold twice.  
Snip double-fold corner.  
Punch opposite corner.

EXTRA STUFF
LESSON 9.5

Distance in Coordinate Geometry

Viki is standing on the corner of Seventh Street and 8th Avenue, and her brother Scott is on the corner of Second Street and 3rd Avenue. To find her shortest sidewalk route to Scott, Viki can simply count blocks. But if Viki wants to know her diagonal distance to Scott, she would need the Pythagorean Theorem to measure across blocks.

You can think of a coordinate plane as a grid of streets with two sets of parallel lines running perpendicular to each other. Every segment in the plane that is not in the x- or y-direction is the hypotenuse of a right triangle whose legs are in the x- and y-directions. So you can use the Pythagorean Theorem to find the distance between any two points on a coordinate plane.

Investigation 1
The Distance Formula

In Steps 1 and 2, find the length of each segment by using the segment as the hypotenuse of a right triangle. Simply count the squares on the horizontal and vertical legs, then use the Pythagorean Theorem to find the length of the hypotenuse.

Step 1
Copy graphs a–d from the next page onto your own graph paper. Use each segment as the hypotenuse of a right triangle. Draw the legs along the grid lines. Find the length of each segment.

LESSON OBJECTIVES
- Discover the Pythagorean relationship on a coordinate plane (the distance formula)
- Derive the equation of a circle from the distance formula
- Use the distance formula to solve problems
- Develop problem-solving skills and cooperative behavior

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Step 2 Graph each pair of points, then find the distances between them.

a. \((1, 7), (11, 1)\)

b. \((9, 6), (3, 10)\)

What if the points are so far apart that it’s not practical to plot them? For example, what is the distance between the points \(A(15, 34)\) and \(B(42, 70)\)? A formula that uses the coordinates of the given points would be helpful. To find this formula, you first need to find the lengths of the legs in terms of the \(x\)- and \(y\)-coordinates. From your work with slope triangles, you know how to calculate horizontal and vertical distances.

Step 3 Write an expression for the length of the horizontal leg using the \(x\)-coordinates.

Step 4 Write a similar expression for the length of the vertical leg using the \(y\)-coordinates.

Step 5 Use your expressions from Steps 3 and 4, and the Pythagorean Theorem, to find the distance between points \(A(15, 34)\) and \(B(42, 70)\).

Step 6 Generalize what you have learned about the distance between two points in a coordinate plane. Copy and complete the conjecture below.

**Distance Formula**

The distance between points \(A(x_1, y_1)\) and \(B(x_2, y_2)\) is given by

\[
(AB)^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2 \quad \text{or} \quad AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}
\]

Let’s look at an example to see how you can apply the distance formula.

Step 1 Students used triangles like this earlier in finding slopes of lines. You might provide The Distance Formula worksheet so students aren’t tempted to write in their books. You might use pair share within groups or jigsaw between groups to save time.

Step 2 As needed, encourage students to use right triangles so they can see the distance formula as a special application of the Pythagorean Theorem. Pair share works well for this step also.

Step 3 Students may want to discuss how to handle negative distances. Some students may want to take the absolute value so that all distances are positive. Others may be comfortable with negative values as a measure of directed distance. Still other students may advocate always subtracting the smaller \(x\)-coordinate from the larger so the distance is always positive.

Step 4 Subtract one \(x\)-coordinate from the other.

Step 5 Subtract one \(y\)-coordinate from the other.

Step 6 Students may remember the distance formula from algebra I, perhaps using \(d\) instead of \(AB\).
EXAMPLE A

Because the distance formula is derived directly from the Pythagorean Theorem, the book begins with the $a^2 + b^2 = c^2$ version of the distance formula. You could also show that you can start with $AB = \sqrt{(x_1 - x_2)^2 + (y_2 - y_1)^2}$.

[Alert] If students are using calculators, they can easily get incorrect answers if parentheses are omitted or not used correctly.

EXAMPLE B

Students may benefit from being reminded that an equation for a geometric figure is satisfied by the coordinates of a point if and only if the point lies on the figure.

Guiding Investigation 2

Step 1 [Alert] Students may inattentively think of diameter instead of radius.

Step 1a

Step 1b

Step 1c

Step 2 As needed, encourage students to use the model in Example B.

After doing this investigation, you may want to work with another example. [Ask] “What is the equation for a circle with center $(2, -3)$ and radius 4?” $(x - 2)^2 + (y + 3)^2 = 16$

Investigation 2

The Equation of a Circle

Find equations for a few more circles and then generalize the equation for any circle with radius $r$ and center $(h, k)$.

Given its center and radius, graph each circle on graph paper.

a. Center $= (1, -2)$, $r = 8$

b. Center $= (0, 2)$, $r = 6$

c. Center $= (-3, -4)$, $r = 10$

Select any point on each circle; label it $(x, y)$. Use the distance formula to write an equation expressing the distance between the center of each circle and $(x, y)$.

Copy and complete the conjecture for the equation of a circle.

Equation of a Circle

The equation of a circle with radius $r$ and center $(h, k)$ is

$$(x - h)^2 + (y - k)^2 = r^2$$

SHARING IDEAS

Students may have come up with a variety of distance formulas. For example, they may have either $x_2 - x_1$ or $y_2 - y_1$ first, or they might have $x_1 - x_2$ in place of $x_2 - x_1$ or $y_1 - y_2$ in place of $y_2 - y_1$. Through student discussion, elicit the idea that these are all the same formula after squaring and adding.

[Ask] “In Example A, how would you know to substitute the coordinates $(8, 15)$ for $(x, y)$ instead of for $(x_2, y_2)$?” Student experimentation can lead to the realization that it doesn’t matter.

When discussing the equation of a circle, ask whether you could get an equivalent formula by taking the square root, as with the distance formula. Students may say $r = \sqrt{(x - h)^2 + (y - k)^2}$. [Ask] “Should the equation include negative values of $r$, $r = \pm \sqrt{(x - h)^2 + (y - k)^2}$? [No; $r$ is a radius, so it cannot be negative.] The equation with $r^2$ is called the standard form.

Ask students how they might use the equation to graph a circle with center at $(1, -2)$ and radius 8. They need to solve the equation for $y$ in terms of $x$. They’ll get $(y + 2)^2 = 64 - (x - 1)^2$, from
Let’s look at an example that uses the equation of a circle in reverse.

**EXAMPLE C**  
Find the center and radius of the circle \((x + 2)^2 + (y - 5)^2 = 36\).

**Solution**  
Rewrite the equation of the circle in the standard form.

\[
(x - h)^2 + (y - k)^2 = r^2
\]

\[
(x - (-2))^2 + (y - 5)^2 = 6^2
\]

Identify the values of \(h\), \(k\), and \(r\). The center is \((-2, 5)\) and the radius is 6.

**EXERCISES**

In Exercises 1–3, find the distance between each pair of points.

1. \((10, 20), (13, 16)\) 5 units  
2. \((15, 37), (42, 73)\) 45 units  
3. \((-19, -16), (-3, 14)\) 34 units

4. Look back at the diagram of Viki’s and Scott’s locations on page 486. Assume each block is approximately 50 meters long. What is the shortest distance from Viki to Scott to the nearest meter? 354 m

5. Find the perimeter of \(\triangle ABC\) with vertices \(A(2, 4), B(8, 12),\) and \(C(24, 0)\). 52.4 units

6. Determine whether \(\triangle DEF\) with vertices \(D(6, -6), E(39, -12),\) and \(F(24, 18)\) is scalene, isosceles, or equilateral. Isosceles

For Exercises 7 and 8, find the equation of the circle.

7. Center = \((0, 0), r = 4\)  
8. Center = \((2, 0), r = 5\)  

For Exercises 9 and 10, find the radius and center of the circle.

9. \((x - 2)^2 + (y + 5)^2 = 6^2\) center is \((2, -5), r = 6\)  
10. \(x^2 + (y - 1)^2 = 81\) center is \((0, 1), r = 9\)

11. The center of a circle is \((3, -1)\). One point on the circle is \((6, 2)\). Find the equation of the circle. \((x - 3)^2 + (y + 1)^2 = 18\)

12. **Mini-Investigation**  
How would you find the distance between two points in a three-dimensional coordinate system? Investigate and make a conjecture. 

a. What is the distance from the origin \((0, 0, 0)\) to \((2, -1, 3)\)? \(\sqrt{14}\) units

b. What is the distance between \(P(1, 2, 3)\) and \(Q(5, 6, 15)\)? \(\sqrt{176} = 4\sqrt{11}\) units

c. Complete this conjecture:  

If \(A(x_1, y_1, z_1)\) and \(B(x_2, y_2, z_2)\) are two points in a three-dimensional coordinate system, then the distance \(AB\) is  

\[
\sqrt{\left(x_2 - x_1\right)^2 + \left(y_2 - y_1\right)^2 + \left(z_2 - z_1\right)^2}.
\]

Sharing Ideas (continued)  
which they want to take square roots to get \(y + 2 = \sqrt{64 - (x - 1)^2}\) and hence \(y = -2 + \sqrt{64 - (x - 1)^2}\). When they graph this function on a calculator, they’ll get only a semicircle, because in taking the square root they eliminated negative values of \(y + 2\). To graph the other half of the circle, they’ll need to graph the second function \(y = -2 - \sqrt{64 - (x - 1)^2}\). Their graphs still might not look very circular. Suggest that they use a friendly window.

**EXAMPLE C**  
Point out that \((x + 2)^2\) can be rewritten as \((x - (-2))^2\).

**Assessing Progress**  
You can assess students’ familiarity with coordinates of points, circles and their radii, right triangles and their hypotenuses, and the Pythagorean Theorem. Also check students’ ability to work separately with the horizontal and vertical distances.

**Closing the Lesson**  
The distance formula,  

\[
AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2},
\]

and the related equation of a circle, \((x - h)^2 + (y - k)^2 = r^2\), are both derived from the Pythagorean Theorem.

**BUILDING UNDERSTANDING**  
These exercises motivate the need for a formula to find the distance between points when plotting the points is impractical.

**ASSIGNING HOMEWORK**

| Essential | 1–10 |
| Performance assessment | 12 |
| Portfolio | 6 |
| Journal | 17 |
| Group | 11 |
| Review | 13–17 |

**Helping with the Exercises**

**Exercise 3** [Alert] Students may not be careful enough in substituting the negative numbers into the distance formula.

**Exercise 11** As needed, encourage students to break apart the problem and consider a diagonal of a horizontal or vertical rectangle as an intermediate step. Or ask whether the distance could be a space diagonal of an imaginary box.
13. Find the coordinates of $A$.

14. $k = \frac{5}{\sqrt{3}}$, $m = \frac{3}{\sqrt{3}}$. $k = \sqrt{2}, m = \sqrt{6}$

15. The large triangle is equilateral. Find $x$ and $y$.

16. Antonio is a biologist studying life in a pond. He needs to know how deep the water is. He notices a water lily sticking straight up from the water, whose blossom is 8 cm above the water’s surface. Antonio pulls the lily to one side, keeping the stem straight, until the blossom touches the water at a spot 40 cm from where the stem first broke the water’s surface. How is Antonio able to calculate the depth of the water? What is the depth? $\text{Ans}.$ 96 cm

7. $C’U’R’T’$ is the image of $CURT$ under a rotation transformation. Copy the polygon and its image onto patty paper. Find the center of rotation and the measure of the angle of rotation. Explain your method.

The angle of rotation is approximately 77°. Connect two pairs of corresponding points. Construct the perpendicular bisector of each segment. The point where the perpendicular bisectors meet is the center of rotation.

**IMPROVING YOUR VISUAL THINKING SKILLS**

**The Spider and the Fly**

(Attributed to the British puzzlist Henry E. Dudeney, 1857–1930)

In a rectangular room, measuring 30 by 12 by 12 feet, a spider is at point $A$ on the middle of one of the end walls, 1 foot from the ceiling. A fly is at point $B$ on the center of the opposite wall, 1 foot from the floor. What is the shortest distance that the spider must crawl to reach the fly, which remains stationary? The spider never drops or uses its web, but crawls fairly.

The shortest path measures 40 feet.
Ladder Climb

Suppose a house painter rests a 20-foot ladder against a building, then decides the ladder needs to rest 1 foot higher against the building. Will moving the ladder 1 foot toward the building do the job? If it needs to be 2 feet lower, will moving the ladder 2 feet away from the building do the trick? Let’s investigate.

Activity

Climbing the Wall

Sketch a ladder leaning against a vertical wall, with the foot of the ladder resting on horizontal ground. Label the sketch using \( y \) for height reached by the ladder and \( x \) for the distance from the base of the wall to the foot of the ladder.

Step 1

Write an equation relating \( x, y \), and the length of the ladder and solve it for \( y \). You now have a function for the height reached by the ladder in terms of the distance from the wall to the foot of the ladder. Enter this equation into your calculator.

Step 2

Before you graph the equation, think about the settings you’ll want for the graph window. What are the greatest and least values possible for \( x \) and \( y \)? Enter reasonable settings, then graph the equation.

Step 3

Describe the shape of the graph.

Step 4

Trace along the graph, starting at \( x = 0 \). Record values (rounded to the nearest 0.1 unit) for the height reached by the ladder when \( x = 3, 6, 9, \) and 12. If you move the foot of the ladder away from the wall 3 feet at a time, will each move result in the same change in the height reached by the ladder? Explain.

Find the value for \( x \) that gives a \( y \)-value approximately equal to \( x \). How is this value related to the length of the ladder? Sketch the ladder in this position. What angle does the ladder make with the floor? Should you lean a ladder against a wall in such a way that \( x \) is greater than \( y \)? Explain. How does your graph support your explanation?

Step 5

When \( x = y \) (at \( x = 14 \), the length of the ladder divided by \( \sqrt{2} \)), the ladder forms a 45° angle with the floor and the wall.

Step 6

When \( x = y \), should you lean a ladder against a wall in such a way that \( x \) is greater than \( y \)? Explain. How does your graph support your explanation?

LESSON OBJECTIVES

- Create an algebraic model for the ladder problem
- Review graphing an equation
- Apply the Pythagorean Theorem to understand how rates of change vary

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PLANNING

LESSON OUTLINE

One day:
- 25 min Activity
- 20 min Sharing and Closing

MATERIALS

- rulers
- graphing calculators

TEACHING

[ESL] Do the trick means “accomplish the task.”

Guiding the Activity

Step 2 Ask students to conjecture whether the amount of change will always be the same and, if not, what it depends on. As needed, help students see where the maximum value of each variable occurs (on an axis).

Step 3 Unless students happened to choose a friendly window in Step 2, they may say that the graph is part of a parabola rather than an arc of a circle.

Step 5 Suggest that students study a table of these data.

Sharing Ideas

As students present their ideas about Steps 4–6, [Ask] “At what point will a one-foot change in \( x \) result in a one-foot change in \( y \)” [at the instant \( y = x \)]

Closing the Lesson

Rates of change of a function will vary if the graph of the function is curved.

See page 775 for answer to Step 6.

EXPLORATION Ladder Climb 491
In Chapter 6, you discovered a number of properties that involved right angles in and around circles. In this lesson you will use the conjectures you made, along with the Pythagorean Theorem, to solve some challenging problems. Let’s review two of the most useful conjectures.

**Tangent Conjecture:** A tangent to a circle is perpendicular to the radius drawn to the point of tangency.

**Angles Inscribed in a Semicircle Conjecture:** Angles inscribed in a semicircle are right angles.

Here are two examples that use these conjectures along with the Pythagorean Theorem.

**EXAMPLE A**

TA is tangent to circle N at A. TA = 12√3 cm. Find the area of the shaded region.

**Solution**

TA is tangent at A, so ∠TAN is a right angle and ΔTAN is a 30°-60°-90° triangle. The longer leg is 12√3 cm, so the shorter leg (also the radius of the circle) is 12 cm. The area of the entire circle is 144π cm². The area of the shaded region is \(\frac{3}{6}(144\pi)\), or 120π cm².

**EXAMPLE B**

AB = 6 cm and BC = 8 cm. Find the area of the circle.

**Solution**

Inscribed angle ABC is a right angle, so ΔABC is a semicircle and AC is a diameter. By the Pythagorean Theorem, if AB = 6 cm and BC = 8 cm, then AC = 10 cm. Therefore the radius of the circle is 5 cm and the area of the circle is 25π cm².

**Closing the Lesson**

Much of the power of geometry comes from combining different conjectures. For example, using what we know about tangent segments, arc lengths, the Pythagorean Theorem, and special right triangles can help us solve a variety of problems.
In Exercises 1–4, find the area of the shaded region in each figure. Assume lines that appear tangent are tangent at the labeled points.

1. \[ OD = 24 \text{ cm} \]
   \[ 456\pi \text{ cm}^2 \]

2. \[ HT = 8\sqrt{3} \text{ cm} \]
   \[ (32\pi - 32\sqrt{3}) \text{ cm}^2 \]

3. \[ HA = 8\sqrt{3} \text{ cm} \]
   \[ (64\pi/3 - 16\sqrt{3}) \text{ cm}^2 \]

4. \[ HO = 8\sqrt{3} \text{ cm} \]
   \[ (64\pi/3 - 64\sqrt{3}) \text{ cm}^2 \]

5. A 3-meter-wide circular track is shown at right. The radius of the inner circle is 12 meters. What is the longest straight path that stays on the track? (In other words, find \( AB \).) \[ 60 \text{ cm} \]

6. An annulus has a 36 cm chord of the outer circle that is also tangent to the inner concentric circle. Find the area of the annulus. \[ 324\pi \text{ cm}^2 \]

7. In her latest expedition, Ertha Diggs has uncovered a portion of circular, terra-cotta pipe that she believes is part of an early water drainage system. To find the diameter of the original pipe, she lays a meterstick across the portion and measures the length of the chord at 48 cm. The depth of the portion from the midpoint of the chord is 6 cm. What was the pipe’s original diameter? \[ 102 \text{ cm} \]

8. **APPLICATION** A machinery belt needs to be replaced. The belt runs around two wheels, crossing between them so that the larger wheel turns the smaller wheel in the opposite direction. The diameter of the larger wheel is 36 cm, and the diameter of the smaller is 24 cm. The distance between the centers of the two wheels is 60 cm. The belt crosses 24 cm from the center of the smaller wheel. What is the length of the belt? \[ 230 \text{ cm} \]

9. A circle of radius 6 has chord \( AB \) of length 6. If point \( C \) is selected randomly on the circle, what is the probability that \( \triangle ABC \) is obtuse? \[ \frac{5}{6} \]

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**LESSON OBJECTIVE**

- Apply the Pythagorean relationship to problems involving circles

**BUILDING UNDERSTANDING**

You might have several groups present their solutions to a few of the exercises. Distribute transparencies and pens to help students prepare their presentations.

**ASSIGNING HOMEWORK**

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**Helping with the Exercises**

**Exercise 3** If students are having difficulty, wonder aloud whether there’s a way to draw auxiliary lines to create right triangles, perhaps special right triangles.

**Exercise 4** As needed, remind students that the area of a segment is the difference between the area of a sector and the area of a triangle.

**Exercise 7** Some students may stop after finding the radius instead of finding the diameter.

**Exercise 8** The answer in terms of \( \pi \) is \( (40\pi + 60\sqrt{3}) \text{ cm} \).

**Exercise 9** As needed, [Ask] "Does a chord that’s congruent to a radius remind you of anything?" [Mark the radius six times around a circle to form a hexagon.] "What happens if point \( C \) falls in the arcs intercepted by the various sides of the inscribed regular hexagon?" [The arc on the hexagon side opposite side \( AB \) would be the only one where the third vertex would create an acute triangle.]
In Exercises 10 and 11, each triangle is equilateral. Find the area of the inscribed circle and the area of the circumscribed circle. How many times greater is the area of the circumscribed circle than the area of the inscribed circle?

10. \(AB = 6\) cm
   - Inscribed circle: \(3\pi\) cm\(^2\).
   - Circumscribed circle: \(12\pi\) cm\(^2\). The area of the circumscribed circle is four times as great as the area of the inscribed circle.

11. \(DE = 2\sqrt{3}\) cm
   - Inscribed circle: \(\pi\) cm\(^2\).
   - Circumscribed circle: \(4\pi\) cm\(^2\). The area of the circumscribed circle is four times as great as the area of the inscribed circle.

12. The Gothic arch is based on the equilateral triangle. If the base of the arch measures 80 cm, what is the area of the shaded region? 3931 cm\(^2\)

13. Each of three circles of radius 6 cm is tangent to the other two, and they are inscribed in a rectangle, as shown. What is the height of the rectangle? 76 cm

14. Sector \(ARC\) has a radius of 9 cm and an angle that measures 80\(^\circ\). When sector \(ARC\) is cut out and \(AR\) and \(RC\) are taped together, they form a cone. The length of \(AC\) becomes the circumference of the base of the cone. What is the height of the cone? \(\sqrt{77}\) cm

15. APPLICATION Will plans to use a circular cross section of wood to make a square table. The cross section has a circumference of 336 cm. To the nearest centimeter, what is the side length of the largest square that he can cut from it? 76 cm

16. Find the coordinates of point \(M\).
   - \((-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})\)

17. Find the coordinates of point \(K\).
   - \((-\frac{\sqrt{3}}{2}, -\frac{1}{2})\)
Review

9.5 18. Find the equation of a circle with center (3, 3) and radius 6. \((x - 3)^2 + (y - 3)^2 = 36\)

9.5 19. Find the radius and center of a circle with the equation 
x^2 + y^2 - 2x + 1 = 100. \(\text{center} = (1, 0), r = 10\)

6.3 20. Construction Construct a circle and a chord in a circle. With compass and straightedge, construct a second chord parallel and congruent to the first chord. Explain your method.

2.6 21. Explain why the opposite sides of a regular hexagon are parallel.

2.3 22. Find the rule for this number pattern:
\[
\begin{align*}
1 & \cdot 3 - 3 = 4 \cdot 0 \\
2 & \cdot 4 - 3 = 5 \cdot 1 \\
3 & \cdot 5 - 3 = 6 \cdot 2 \\
4 & \cdot 6 - 3 = 7 \cdot 3 \\
5 & \cdot 7 - 3 = 8 \cdot 4 \\
\vdots \\
\end{align*}
\]
\[
\left( \frac{n}{3} \right) - \left( \frac{n}{3} \right) = \left( \frac{n}{3} \right) \cdot \left( \frac{n}{3} - 1 \right)
\]
\[
\left( n + 1 \right) \left( n + 2 \right) - 3 = \left( n + 3 \right) \cdot \left( n - 1 \right)
\]

6.2 23. Application Felice wants to determine the diameter of a large heating duct. She places a carpenter’s square up to the surface of the cylinder, and the length of each tangent segment is 10 inches.

a. What is the diameter? Explain your reasoning.
b. Describe another way she can find the diameter of the duct.
   Possible answer: Measure the circumference with string and divide by \(\pi\).

23a. Because a carpenter’s square has a right angle and both radii are perpendicular to the tangents, a square is formed. The radius is 10 in., therefore the diameter is 20 in.

IMPROVING YOUR REASONING SKILLS

Reasonable 'Rithmetic I

Each letter in these problems represents a different digit.

1. What is the value of \(B\)?
\[
\begin{array}{ccc}
3 & 7 & 2 \\
3 & 8 & 4 \\
+ & 9 & B \\
\end{array}
\]
\[
\begin{array}{ccc}
C & 7 & C \\
D & D & F \\
E & D & D \\
\end{array}
\]

2. What is the value of \(J\)?
\[
\begin{array}{ccc}
E & F & 6 \\
\times & & D \\
\end{array}
\]
\[
\begin{array}{ccc}
D & D & F \\
J & E & D \\
H & G & E \\
\end{array}
\]

If students are having difficulty, suggest that they make a table of the letters, listing what they can say about each letter.

1. \(A = 0\) and \(C = 1\), so \(B = 5\).
2. \(D = 2\); \(7F + 4\) ends in \(F\), so \(F = 1\) or 6. Trying each possibility leads to \(J = 6\).

EXTENSION

Ask students to show that the shaded area in this figure is equal to the area of the triangle. See Leonardo’s Dessert—No Pi by Herbert Wills for further study of problems like this.
If 50 years from now you’ve forgotten everything else you learned in geometry, you’ll probably still remember the Pythagorean Theorem. (Though let’s hope you don’t really forget everything else!) That’s because it has practical applications in the mathematics and science that you encounter throughout your education.

It’s one thing to remember the equation \( a^2 + b^2 = c^2 \). It’s another to know what it means and to be able to apply it. Review your work from this chapter to be sure you understand how to use special triangle shortcuts and how to find the distance between two points in a coordinate plane.

**EXERCISES**

For Exercises 1–4, measurements are given in centimeters.

1. \( x = \frac{7}{2} \text{ cm} \)

2. \( AB = \frac{5}{2} \text{ cm} \)

3. Is \( \triangle ABC \) an acute, obtuse, or right triangle? \( \text{obtuse} \)

4. The solid is a rectangular prism. \( AB = \frac{5}{2} \text{ cm} \)

5. Find the coordinates of point \( U \). \( \left( \frac{\sqrt{3}}{2}, \frac{1}{2} \right) \)

6. Find the coordinates of point \( V \). \( \left( \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right) \)

7. What is the area of the triangle? \( 200\sqrt{3} \text{ cm}^2 \)

8. The area of this square is 144 cm\(^2\). Find \( d = 12\sqrt{2} \text{ cm} \)

9. What is the area of trapezoid \( ABCD \)? \( 246 \text{ cm}^2 \)
10. The arc is a semicircle. What is the area of the shaded region? 
\[ 72\pi \text{ in}^2 \]

11. Rays TA and TB are tangent to circle O at A and B respectively, and BT = 6\sqrt{3} \text{ cm}. What is the area of the shaded region? 
\[ 24\pi \text{ cm}^2 \]

12. The quadrilateral is a square, and QE = 2\sqrt{2} \text{ cm}. What is the area of the shaded region? 
\[ (2\pi - 4) \text{ cm}^2 \]

13. The area of circle Q is 350 cm\(^2\). Find the area of square ABCD to the nearest 0.1 cm\(^2\). 
\[ 222.8 \text{ cm}^2 \]

14. Determine whether \( \triangle ABC \) with vertices A(3, 5), B(11, 3), and C(8, 8) is an equilateral, isosceles, or isosceles right triangle. 
Isosceles right

15. Sagebrush Sally leaves camp on her dirt bike traveling east at 60 km/hr with a full tank of gas. After 2 hours, she stops and does a little prospecting—with no luck. So she heads north for 2 hours at 45 km/hr. She stops again, and this time hits pay dirt. Sally knows that she can travel at most 350 km on one tank of gas. Does she have enough fuel to get back to camp? If not, how close can she get? 
No. The closest she can come to camp is 10 km.

16. A parallelogram has sides measuring 8.5 cm and 12 cm, and a diagonal measuring 15 cm. Is the parallelogram a rectangle? If not, is the 15 cm diagonal the longer or shorter diagonal? 
No. 15 cm is the longer diagonal.

17. After an argument, Peter and Paul walk away from each other on separate paths at a right angle to each other. Peter is walking 2 km/hr, and Paul is walking 3 km/hr. After 20 min, Paul sits down to think. After 30 min, Peter stops. Both decide to apologize. How far apart are they? How long will it take them to reach each other if they both start running straight toward each other at 5 km/hr? 
1.4 km; 8.5 min

18. Flora is away at camp and wants to mail her flute back home. The flute is 24 inches long. Will it fit diagonally within a box whose inside dimensions are 12 by 16 by 14 inches? 
Yes

19. To the nearest foot, find the original height of a fallen flagpole that cracked and fell as if hinged, forming an angle of 45 degrees with the ground. The tip of the pole hit the ground 12 feet from its base. 
29 ft
20. You are standing 12 feet from a cylindrical corn-syrup storage tank. The distance from you to a point of tangency on the tank is 35 feet. What is the radius of the tank? \( \approx 45 \) ft

**Technology CONNECTION**

Radio and TV stations broadcast from high towers. Their signals are picked up by radios and TVs in homes within a certain radius. Because Earth is spherical, these signals don’t get picked up beyond the point of tangency.

21. **APPLICATION** Read the Technology Connection above. What is the maximum broadcasting radius from a radio tower 1800 feet tall (approximately 0.34 mile)? The radius of Earth is approximately 3960 miles, and you can assume the ground around the tower is nearly flat. Round your answer to the nearest 10 miles. 50 mi

22. A diver hooked to a 25-meter line is searching for the remains of a Spanish galleon in the Caribbean Sea. The sea is 20 meters deep and the bottom is flat. What is the area of circular region that the diver can explore? \( 707 \) m²

23. What are the lengths of the two legs of a 30°-60°-90° triangle if the length of the hypotenuse is 12\( \sqrt{3} \)? 6\( \sqrt{3} \) and 18

24. Find the side length of an equilateral triangle with an area of 36\( \sqrt{3} \) m². 12 m

25. Find the perimeter of an equilateral triangle with a height of 7\( \sqrt{3} \). 42

26. No. If you reflect one of the right triangles into the center piece, you’ll see that the area of the kite is almost half again as large as the area of each of the other triangles.

Or students might compare areas by assuming the short leg of the 30°-60°-90° triangle is 1. The area of each triangle is then \( \frac{\sqrt{3}}{2} \) and the area of the kite is \( 3 - \sqrt{3} \).
28. One of the sketches below shows the greatest area that you can enclose in a right-angled corner with a rope of length $s$. Which one? Explain your reasoning.

\[ \begin{align*}
\text{A triangle} & \quad \text{A square} & \quad \text{A quarter-circle} \\
\end{align*} \]

29. A wire is attached to a block of wood at point $A$. The wire is pulled over a pulley as shown. How far will the block move if the wire is pulled 1.4 meters in the direction of the arrow? 1.6 m

**MIXED REVIEW**

30. **Construction** Construct an isosceles triangle that has a base length equal to half the length of one leg.

31. In a regular octagon inscribed in a circle, how many diagonals pass through the center of the circle? In a regular nonagon? a regular 20-gon? What is the general rule?

32. A bug clings to a point two inches from the center of a spinning fan blade. The blade spins around once per second. How fast does the bug travel in inches per second? In Exercises 33–40, identify the statement as true or false. For each false statement, explain why it is false or sketch a counterexample.

33. The area of a rectangle and the area of a parallelogram are both given by the formula $A = bh$, where $A$ is the area, $b$ is the length of the base, and $h$ is the height. **true**

34. When a figure is reflected over a line, the line of reflection is perpendicular to every segment joining a point on the original figure with its image. **true**

35. In an isosceles right triangle, if the legs have length $x$, then the hypotenuse has length $x\sqrt{2}$. False. The hypotenuse is of length $\frac{x}{\sqrt{2}}$.

36. The area of a kite or a rhombus can be found by using the formula $A = \frac{1}{2}d_1d_2$, where $A$ is the area and $d_1$ and $d_2$ are the lengths of the diagonals. **true**

37. If the coordinates of points $A$ and $B$ are $(x_1, y_1)$ and $(x_2, y_2)$, respectively, then $AB = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$. False; $AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

38. A glide reflection is a combination of a translation and a rotation. False. A glide reflection is a combination of a translation and a reflection.

Exercise 31 Encourage students to make a table and look for a pattern.

31. 4; 0; 10. The rule is $\frac{n}{2}$ if $n$ is even, but 0 if $n$ is odd.
7.4 39. Equilateral triangles, squares, and regular octagons can be used to create monohedral tessellations. False. Equilateral triangles, squares, and regular hexagons can be used to create monohedral tessellations.

9.3 40. In a 30°-60°-90° triangle, if the shorter leg has length $x$, then the longer leg has length $x\sqrt{3}$ and the hypotenuse has length $2x$. True

In Exercises 41–46, select the correct answer.

9.1 41. The hypotenuse of a right triangle is always □.
A. opposite the smallest angle and is the shortest side.
B. opposite the largest angle and is the shortest side.
C. opposite the smallest angle and is the longest side.
D. opposite the largest angle and is the longest side.

8.2 42. The area of a triangle is given by the formula $A = \frac{1}{2}bh$, where $A$ is the area, $b$ is the length of the base, and $h$ is the height. □
A. $A = bh$
B. $A = \frac{1}{2}bh$
C. $A = 2bh$
D. $A = b^2h$

9.2 43. If the lengths of the three sides of a triangle satisfy the Pythagorean equation, then the triangle must be a(n) △ triangle. □
A. right
B. acute
C. obtuse
D. scalene

7.2 44. The ordered pair rule $(x, y) \rightarrow (y, x)$ is a □.
A. reflection over the $x$-axis
B. reflection over the $y$-axis
C. reflection over the line $y = x$
D. rotation 90° about the origin

7.3 45. The composition of two reflections over two intersecting lines is equivalent to □.
A. a single reflection
B. a translation
C. a rotation
D. no transformation

8.7 46. The total surface area of a cone is equal to □, where $r$ is the radius of the circular base and $l$ is the slant height. □
A. $\pi r^2 + 2\pi r$
B. $\pi rl$
C. $\pi rl + 2\pi r$
D. $\pi rl + \pi r^2$

5.7 47. Create a flowchart proof to show that the diagonal of a rectangle divides the rectangle into two congruent triangles.

47. 
1. $ABCD$ is a rectangle
   Given
2. $ABCD$ is a parallelogram
   Definition of rectangle
3. $\angle D = \angle B$
   Definition of rectangle
4. $DA \parallel CB$
   Definition of parallelogram
5. $\angle DAC = \angle BCA$
   AIA Conjecture
6. $\triangle ABC \cong \triangle DCA$
   SAA Congruence Conjecture
7. $AC = AC$
   Same segment
1.2 48. Copy the ball positions onto patty paper.
   a. At what point on the S cushion should a player aim so that the cue ball bounces off and strikes the 8-ball? Mark the point with the letter A.
   b. At what point on the W cushion should a player aim so that the cue ball bounces off and strikes the 8-ball? Mark the point with the letter B.

8.2 49. Find the area and the perimeter of the trapezoid. 34 cm²; $22 + 4\sqrt{2} \approx 27.7$ cm

8.6 50. Find the area of the shaded region. $\frac{40\pi}{3}$ cm²

9.1 51. An Olympic swimming pool has length 50 meters and width 25 meters. What is the diagonal distance across the pool?

9.1 53. The box below has dimensions 25 cm, 36 cm, and x cm. The diagonal shown has length 65 cm. Find the value of x. 48 cm

51. about 55.9 m²
52. about 61.5 cm²

TAKE ANOTHER LOOK

1. Use geometry software to demonstrate the Pythagorean Theorem. Does your demonstration still work if you use a shape other than a square—for example, an equilateral triangle or a semicircle?
2. Find Elisha Scott Loomis’s Pythagorean Proposition and demonstrate one of the proofs of the Pythagorean Theorem from the book.

Take Another Look

1. Demonstrations should include shapes other than a square. (Any regular ploygon can be used. In fact, any three similar figures will work. Students will study similar figures in Chapter 11.)
2. Demonstrations will vary.
3. The small square in the center has sides $b - a$, the slanted square has area $\frac{ab}{2}$. The equation $c^2 = (b - a)^2 + \left(\frac{ab}{2}\right)^2$ simplifies to $c^2 = a^2 + b^2$.

4. One possible proof: Given $\triangle ABC$ with $BC = x$, $AC = x\sqrt{3}$, and $AB = 2x$, construct $30^\circ$-$60^\circ$-$90^\circ$ right triangle $DEF$ with right angle $F$, $30^\circ$ angle $D$, and $EF = x$. $DF = x\sqrt{3}$ and $DE = 2x$, by the $30^\circ$-$60^\circ$-$90^\circ$ Triangle Conjecture. $\triangle ABC \cong \triangle DEF$ by SSS. $\angle C \equiv \angle F$ and is a right angle, $\angle A \equiv \angle D$ and is a $30^\circ$ angle, and $\angle B \equiv \angle E$ and is a $60^\circ$ angle.

5. Starting with an isosceles right triangle, use geometry software or a compass and straightedge to start a right triangle like the one shown. Continue constructing right triangles on the hypotenuse of the previous triangle at least five more times. Calculate the length of each hypotenuse and leave them in radical form.

### Assessing What You’ve Learned

- **UPDATE YOUR PORTFOLIO** Choose a challenging project, Take Another Look activity, or exercise you did in this chapter and add it to your portfolio. Explain the strategies you used.
- **ORGANIZE YOUR NOTEBOOK** Review your notebook and your conjecture list to be sure they are complete. Write a one-page chapter summary.
- **WRITE IN YOUR JOURNAL** Why do you think the Pythagorean Theorem is considered one of the most important theorems in mathematics?
- **WRITE TEST ITEMS** Work with group members to write test items for this chapter. Try to demonstrate more than one way to solve each problem.
- **GIVE A PRESENTATION** Create a visual aid and give a presentation about the Pythagorean Theorem.

### Facilitating Self-Assessment

As part of a final grade, you might ask students to present visual demonstrations or animations of various proofs of the Pythagorean Theorem. (See Take Another Look activities 1–3.) If this presentation replaces the chapter test, you can give a unit exam after working through the mixed review.