Overview

In Discovering Advanced Algebra, students study mathematical functions modeling real-world problems. Chapter 4 is at the core of that study. Here, the abstract idea of a function grows out of students' earlier experiences with linear equations and graphing.

This chapter considers both linear and nonlinear functions and how changing a function’s expression transforms its graph. Students also encounter an even further abstraction—the idea of a relation—and they study equations and graphs of ellipses.

This chapter begins with graphs in Lesson 4.1. Lesson 4.2 makes the distinction between relations and functions as it introduces function notation. Students look at translations of linear functions in Lesson 4.3. Lesson 4.4 presents the family of quadratic functions as transformations of the function $y = x^2$ and emphasizes the vertex as a key to writing these equations from a graph or graphing the equations. Lesson 4.5 uses another transformation, reflection, to examine the square root family, with parent function $y = \sqrt{x}$. In the exploration students see a rotation as a composition of two reflections. Students learn about dilations to help explore the absolute-value family of functions in Lesson 4.6. Lesson 4.7 considers transformations of the circle and ellipse family of relations. Lesson 4.8 looks at compositions of functions.

The Mathematics

Relations and Functions

A relation can be thought of as a two-column table of numbers. The items in the first column make up the relation’s domain; the second column is its range.

One way you can represent relations is with graphs. You can regard each table row of two numbers as the coordinates of a point on a plane. The relation’s graph contains all of those points. If a graph consists of disconnected points, the relation is discrete; otherwise, it’s continuous.

The most common kind of relation is a function, in which no number appears twice in the first column. A function can therefore be thought of as “taking” each number in the first column to the corresponding number in the second. For functions, the two columns may be called input and output or independent variable and dependent variable.

In the case of a function, the equation relating the variables is often called the function’s rule. For example, the equation $y = x^2 + 2$ tells how variable $y$ depends on variable $x$. To emphasize that $y$ is a function of $x$, this rule might also be written $f(x) = x^2 + 2$.

Building Complex Relations

To understand and graph complicated relations, it’s often useful to see how they are made up from simpler relations. For example, relations may be transformations of simpler relations. Transformations are usually thought of as motions of a graph. This chapter addresses three ways in which graphs might be transformed.

One kind of transformation is the translation (shift). A horizontal translation of a graph to the right is like replacing the $x$ in the equation with $(x - h)$. For example, $y = (x - 2)^2$ represents a translation of the graph of $y = x^2$ to the right 2 units. A vertical translation upward is like replacing the $y$ in the equation with $(y - k)$. The graph of the function with equation $y = 3 = x^2$ is a translation of the graph of $y = x^2$ up 3 units.
Another kind of transformation is a reflection across an axis. A horizontal reflection (across the vertical axis) corresponds to multiplying $x$ by $-1$, and a vertical reflection corresponds to multiplying $y$ by $-1$. For example, $y = (-x)^3$ reflects the graph of $y = x^3$ horizontally, across the $y$-axis. And $-y = x^3$ reflects the same graph vertically, across the $x$-axis.

A relation can also be dilated (stretched from or shrunk toward an axis). A horizontal stretch corresponds to dividing $x$ by a factor that is greater than 1, and a vertical stretch to dividing $y$ by a factor greater than 1. The equation $\left(\frac{1}{5}\right)^2 + \left(\frac{1}{7}\right)^2 = 1$ represents the ellipse obtained by stretching the unit circle $x^2 + y^2 = 1$ horizontally by a factor of 5 and vertically by a factor of 7. To shrink a graph, the dilation factor is less than 1.

If a complex relation is a function, it may be the composition of simpler functions. You can think of the composition of functions as one function followed by the other. For example, if $f(x) = (x - 4)^2$ and $g(x) = 3x$, then the composition $g(f(x))$ is $y = 3(x - 4)^2$, which is $y = 3x(x - 4)^2$.

### Using This Chapter

Lessons 4.4 and 4.5 can be covered in one day if the number of exercises assigned is limited. The investigation in Lesson 4.8 is optional; the exercises include real-world applications that support the lesson well.

### Materials

- graph paper
- motion sensors
- geometry software
- string
- small weights
- stopwatches or watches with second hand
- tape measures or metersticks
- small mirrors
American artist Benjamin Edwards (b. 1970) used a digital camera to collect images of commercial buildings for this painting, Convergence. He then projected all the images in succession on a 97-by-146-inch canvas, and filled in bits of each one. The result is that numerous buildings are transformed into one busy impression—much like the impression of seeing many things quickly out of the corner of your eye when driving through a city.

There are about 250 sites featured in the painting. Edwards aims to capture a look at suburban sprawl; he intends for the painting to be overwhelming and difficult to look at. [Ask] “What do you think is the artist’s opinion of suburban sprawl?” [Sample answer: It is too busy. Developers try to put too many strip malls and superstores into a small, peaceful space, and it ends up being overwhelming.]

[Ask] “This chapter is partly about transformations. How does this painting represent a transformation?” [It consists of real images that have been transformed into something different and almost unrecognizable. The artist has translated hundreds of images into one place.] “What images do you recognize?” [building in the upper-right corner, chunks of brick, white fence on the left] “What do you think other parts of the painting represent?” [Sample answers: The section at the bottom represents a parking lot, with the lines representing the chaos of traffic. The black splotches represent bushes. The white dot at the top and just right of center represents the sun.]

In this chapter you will
- interpret graphs of functions and relations
- review function notation
- learn about the linear, quadratic, square root, absolute-value, and semicircle families of functions
- apply transformations—translations, reflections, and dilations—to the graphs of functions and relations
- transform functions to model real-world data

- Describe a graph as discrete or continuous and identify the independent and dependent variables, the intercepts, and the rates of change
- Draw a qualitative graph from a context scenario and create a context scenario given a qualitative graph
- Define function, domain, and range, and use function notation
- Distinguish conceptually and graphically between functions and relations
- Study linear, quadratic, absolute-value, square root, and semicircle families of functions
- Use $\sqrt{1-x^2}$ and piecewise-constructed functions defined over bounded intervals to explore relationships between transformations and their equations and graphs
- See how translations, reflections, stretches, and compressions of the graphs of these functions and of the unit circle affect their equations
- Explore compositions of transformations graphically and numerically in real-world contexts
Solving Equations

When you evaluate an expression, you must follow the order of operations: parentheses, exponents, multiplication/division, addition/subtraction. When you solve equations, it is often helpful to think of reversing this order of operations in order to “undo” all that was done to the variable.

The absolute value of a number is its distance from zero on the number line. The equation \( |x| = 5 \) has two solutions, either \( x = 5 \) or \( x = -5 \), because both 5 and -5 are five units from zero on the number line.

**Example A**

Solve \( 5|a - 2| = 12 \).

**Solution**

Consider the operations performed on \( a \). First subtract 2 from \( a \), then take the absolute value of the result, and finally, multiply by 5. To solve this equation, you can undo these steps in reverse order.

\[
\begin{align*}
5|a - 2| &= 12 & \text{Original equation.} \\
\frac{1}{5} \cdot 5|a - 2| &= \frac{1}{5} \cdot 12 & \text{Multiply by the reciprocal of 5 (to undo multiplying by 5).} \\
|a - 2| &= \frac{12}{5} = 2.4 & \text{Multiply and change to decimal form.}
\end{align*}
\]

To undo the absolute value, you’ll need to consider two possibilities. The value \((a - 2)\) is 2.4 units from 0 on the number line, so \((a - 2)\) might equal 2.4 or \((a - 2)\) might equal -2.4.

\[
\begin{align*}
a - 2 &= 2.4 & a - 2 &= -2.4 & \text{Unde the absolute value.} \\
a &= 4.4 & a &= -0.4 & \text{Add 2 to undo subtracting 2.}
\end{align*}
\]

Check both answers to verify that they satisfy the original equation.

\[
\begin{align*}
5|4.4 - 2| &= 12 & 5|0.4 - 2| &= 12 \\
5|2.4| &= 12 & 5|-2.4| &= 12
\end{align*}
\]

Just as there are two solutions to the equation \( |x| = 5 \), there are two solutions to the equation \( x^2 = 25 \). You can take the square root of both sides of an equation if both sides are positive, but be careful! Note that for negative values of \( x \), \( \sqrt{x^2} \neq x \).
For example, \( \sqrt{(-5)^2} \) equals 5, not \( -5 \). An equation that is true for all values of \( x \) is \( \sqrt{x^2} = |x| \). Convince yourself of this by substituting some positive and negative values for \( x \) into \( \sqrt{x^2} \).

If you use the absolute value in solving equations with \( x \)-squared, you won’t forget to find both solutions.

**EXAMPLE B**  
Solve \( 8 + 2(b - 6)^2 = 26 \).

**Solution**  
Undo the operations performed on the variable \( b \) in reverse order.

\[
\begin{align*}
8 + 2(b - 6)^2 &= 26 & \text{Original equation.} \\
2(b - 6)^2 &= 18 & \text{Add } -8 \text{ to each side to undo adding } 8. \\
(b - 6)^2 &= 9 & \text{Multiply by } \frac{1}{2} \text{ to undo multiplying by } 2. \\
\sqrt{(b - 6)^2} &= \sqrt{9} & \text{Take the square root of each side to undo squaring.} \\
|b - 6| &= 3 & \text{Use the relationship } \sqrt{x^2} = |x|. \\
b - 6 &= 3 \text{ or } b - 6 &= -3 & \text{Undo the absolute value.} \\
b &= 9 \text{ or } b &= 3 & \text{Add } 6 \text{ to each side to undo subtracting } 6.
\end{align*}
\]

Once again, you should check your answers in the original equation.

If you are solving an equation in which the variable is inside a square root, you can reverse the square root by squaring each side of the equation.

**EXAMPLE C**  
Solve \( \sqrt{c} + 3 = 9 \).

**Solution**  
To solve, undo the operations in reverse order.

\[
\begin{align*}
\sqrt{c} + 3 &= 9 & \text{Square each side to undo the square root.} \\
d + 3 &= 81 & \text{Square.} \\
c &= 78 & \text{Add } -3 \text{ to each side to undo adding } 3.
\end{align*}
\]

You can check this answer mentally to see that it works in the original equation.

**EXERCISES**

1. Identify the first step in solving each of the equations for the variable. (It may be helpful to first identify the order of operations.)
   
   a. \( \frac{2}{3}x - 7 = 15 \)  
   b. \( 3|x + 8| = 21 \)  
   c. \( 2 + 3(x - 1)^2 = 82 \)  
   d. \( \sqrt{y - 8} = 7 \)  
   e. \( |x - 3| + 6 = 1 \)

2. Solve the equations in Exercise 1.

3. Check the answers you found in Exercise 2. Did all of your answers check? Explain.

**LESSON EXAMPLE B**  
[Alert]  
Students might not understand why they should use the absolute value when taking the square root. Remind them that the absolute-value and the square root symbols both indicate nonnegative numbers.

**LESSON EXAMPLE C**  
Students might be shy about squaring, wondering whether they must consider two cases. Compliment their care in thinking about square roots, but point out that only positive values are being squared here. In addition, squares of negatives and positives of the same magnitude are identical.

**EXERCISE NOTES**

**Exercise 3** Check that students are not assuming that they have found a solution before they complete the check. Note the use of question marks above the equal signs in Lesson Example A. If students skip these checks, they might not realize that Exercise 2e does not have a solution.
LE S S O N 4.1

OBJECTIVES

- Identify independent and dependent variables
- Interpret features of a qualitative graph, including rates of change and x- and y-intercepts
- Decide whether a graph (or a function) is discrete or continuous when given a description of the variables
- Draw a qualitative graph from a context scenario and create a context scenario given a qualitative graph
- Distinguish between linear change and nonlinear change

OUTLINE

One day:
10 min Example
15 min Investigation
5 min Discuss Investigation
15 min Exercises

MATERIALS

- Investigation Worksheet, optional
- More Graph Stories (T), optional

ADDITIONAL SUPPORT

- Lesson 4.1 More Practice Your Skills
- Lesson 4.1 Condensed Lessons (in English or Spanish)
- TestCheck worksheets

TEACHING THE LESSON

This lesson reviews many aspects of representing real-world situations with graphs.

ONGOING ASSESSMENT

As students work, check their understanding of real-world connections to increasing or decreasing curves and of discrete and continuous phenomena. You can also see how well they read and write and how well they work with two variables.

Discussing the Lesson

Before students look at the book, you may want to present the haircut scenario and have a discussion about which variable would be independent. Many students will claim that the number of haircuts is the independent variable, especially if you mention it first, as the book does. [Ask] “If you owned a hair salon, how would you determine the cost of a haircut?” Indeed, the number of customers per week may be one of several variables that help determine the price. After students look at the graph in the book, ask [Critical Question] “Why isn’t the y-intercept bigger? Is this a linear relationship?” [Big Idea] It might not be; no matter what the price, it seems that someone is still willing to pay it.

Interpreting Graphs

A picture can be worth a thousand words, if you can interpret the picture. In this lesson you will investigate the relationship between real-world situations and graphs that represent them.

What is the real-world meaning of the graph at right, which shows the relationship between the number of customers getting haircuts each week and the price charged for each haircut?

The number of customers depends on the price of the haircut. So the price in dollars is the independent variable and the number of customers is the dependent variable. As the price increases, the number of customers decreases linearly. As you would expect, fewer people are willing to pay a high price; a lower price attracts more customers.

The slope indicates the number of customers lost for each dollar increase. The x-intercept represents the haircut price that is too high for anyone. The y-intercept indicates the number of customers when haircuts are free.

EXAMPLE

Students at Central High School are complaining that the juice vending machine is frequently empty. Several student council members decide to study this problem. They record the number of cans in the machine at various times during a typical school day and make a graph.

a. Based on the graph, at what times is juice consumed most rapidly?

b. When is the machine refilled? How can you tell?

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c. When is the machine empty? How can you tell?
d. What do you think the student council will recommend to solve the problem?

**Solution**

Each horizontal segment indicates a time interval when juice does not sell. Negative slopes represent when juice is consumed, and positive slopes show when the machine is refilled.

a. The most rapid consumption is pictured by the steep, negative slopes from 11:30 A.M. to 12:30 P.M., and from 3:00 to 3:30 P.M.
b. The machine is completely refilled overnight, again at 10:30 A.M., and again just after school lets out for the day. The machine is also refilled at 12:30 P.M., but only to 75% capacity.
c. The machine is empty from 3:30 to 4:00 P.M., and briefly at about 12:30 P.M.
d. The student council might recommend refilling the machine once more at about 2:00 or 3:00 P.M. in order to solve the problem of its frequently being empty. Refilling the machine completely at 12:30 P.M. may also solve the problem.

**Health**

Many school districts and several states have banned vending machines and the sale of soda pop and junk foods in their schools. Proponents say that schools have a responsibility to promote good health. The U.S. Department of Agriculture already bans the sale of foods with little nutritional value, such as soda, gum, and popcicles, in school cafeterias, but candy bars and potato chips don’t fall under the ban because they contain some nutrients. Although the student council members in the example are interested in solving a problem related to juice consumption, they could also use the graph to answer many other questions about Central High School: When do students arrive at school? What time do classes begin? When is lunch? When do classes let out for the day?

Both the graph of haircut customers and the graph in the example are shown as continuous graphs. In reality, the quantity of juice in the machine can take on only discrete values, because the number of cans must be a whole number. The graph might more accurately be drawn with a series of short horizontal segments, as shown at right. The price of a haircut and the number of customers can also take on only discrete values. This graph might be more accurately drawn with separate points. However, in both cases, a continuous “graph sketch” makes it easier to see the trends and patterns.

**DIFFERENTIATING INSTRUCTION**

**ELL**
- It may be helpful to relate verbal questions to mathematical expressions. For example, “When is the machine empty?” could be asked as, “When is \( y = 0 \)?”
- You may also have students tell a story in their primary language.

**Extra Support**
- Encourage students to give detailed descriptions of graphs rather than giving a quick answer. Guide students to break down the graph into segments and to write a brief description for each part of the graph.

**Advanced**
- Have students look ahead to Chapter 8 or 13 and write situations for the graphs they find there.

**LESSON EXAMPLE**

[Critical Question] “Does the graph indicate any other information about the school?” [Big Idea] Apparently students arrive at school at 7:00 in the morning; classes begin at 8:00; lunch begins at 11:30; classes let out at 3:00. If you have time and your own school has vending machines, suggest that students sketch a graph representing their estimate of the stock in one of these machines.

[Ask] “How would you describe the slopes of the lines representing refills?” [The slopes are very large.]

Although the student council members in the example are interested in solving a problem related to juice consumption, they could also use the graph to answer many other questions about Central High School: When do students arrive at school? What time do classes begin? When is lunch? When do classes let out for the day?

These recycled aluminum cans are waiting to be melted and made into new cans. Although 65% of the United States’ aluminum is currently recycled, 1 million tons are still thrown away each year.

**LESSON 4.1 Interpreting Graphs**

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Guiding the Investigation

This is an exploring concepts investigation.

To add to the variety, you might use the More Graph Stories transparency and ask groups to work on different graphs.

MODIFYING THE INVESTIGATION

Whole Class: Complete Parts 1 and 2 with student input. Then have a student try a story or a diagram and have the class interpret it.

Shortened: Choose either Part 1 or Part 2.

One Step: Go directly to the investigation, without introduction. During the discussion, lead the class in making a table showing relationships between real-world situations and graphs. (See Closing the Lesson.) As needed, go to the example to see whether its solution is consistent with the table.

FACILITATING STUDENT WORK

This creative activity may help deepen students’ understanding of slopes as representing rates of change. It also is an additional attraction to mathematics for students who like to write or be creative. If time is limited, have half the class work on Part 1 and the other half on Part 2.

ASSESSING PROGRESS

Watch students’ interpretation and understanding of curved lines, which frequently represent acceleration or deceleration, and of step functions.

DISCUSSING THE INVESTIGATION

As students share their stories and graphs, ask what units are appropriate for each variable. Suggest that students help communicate their ideas by superimposing a grid on the graph or by labeling points to reference in their story.

Investigation

Graph a Story

Every graph tells a story. Make a graph to go with the story in Part 1. Then invent your own story to go with the graph in Part 2.

Part 1

Sketch a graph that reflects all the information given in this story.

"It was a dark and stormy night. Before the torrents of rain came, the bucket was empty. The rain subsided at daybreak. The bucket remained untouched through the morning until Old Dog Trey arrived as thirsty as a dog. The sun shone brightly through the afternoon. Then Billy, the kid next door, arrived. He noticed two plugs in the side of the bucket. One of them was about a quarter of the way up, and the second one was near the bottom. As fast as you could blink an eye, he pulled out the plugs and ran away."

Part 2

This graph tells a story. It could be a story about a lake, a bathtub, or whatever you imagine. Spend some time with your group discussing the information contained in the graph. Write a story that conveys all of this information, including when and how the rates of change increase or decrease.

Science Connection

Contour maps are a way to graphically represent altitude. Each line marks all of the points that are the same height in feet (or meters) above sea level. Using the distance between two contour lines, you can calculate the rate of change in altitude. These maps are used by hikers, forest fire fighters, and scientists.

[Critical Question] Ask students to identify the dependent and independent variables in each case. Encourage discussion; often the distinction isn’t clear. Welcome challenges to your own ideas, but try to articulate your intuition. In this context, you can also review domain and range.

[Ask] "Do all stories give continuous graphs?" Some of the stories might describe discrete situations; help students see that in those cases continuous graphs are inappropriate.

SUPPORT EXAMPLES

1. Draw three examples of increasing graphs of real-world situations. [Answers will vary.]

2. Give a real-world example of a decreasing continuous graph. [Possible answer: the temperature of a cup of hot water placed in the freezer]
As you interpret data and graphs that show a relationship between two variables, you must always decide which is the independent variable and which is the dependent variable. You should also consider whether the variables are discrete or continuous.

**Exercises**

**Practice Your Skills**

1. Match a description to each graph.

2. Sketch a graph to match each description.

3. For each graph, write a description like those in Exercise 2.

**Closing the Lesson**

The main point of this lesson is that graphs can represent many aspects of real world situations.

**Exercise Notes**

Most of the exercises have more than one correct answer. If you haven’t already been stressing that students’ work should include responses to the question “Why?” even when this question is not actually stated, now is a good time to do so.

For each graph, ask students to label each axis with a quantity (such as time or distance); they need not indicate numerical units. The important factors are which variable is independent, the shape of the graph, and whether the graph is continuous or discrete.

**Exercise 1**

“Graphs a and b are increasing. In which graphs is the rate of growth increasing?”

[a and d] The rate itself is given by the slope; the rate is increasing if the slope is getting more positive or less negative. So even when the slope is negative, it can be increasing, as in 1d, from “more negative” to “less negative.”

**Exercise 2**

An extensive set of activities and exercises for interpreting graphs appears in the book *A Visual Approach to Functions*.
4a. Possible answer: The curve might describe the relationship between the amount of time the ball is in the air and how far away from the ground it is.

4c. Possible answer: Domain: \(0 \leq t \leq 10 \) s; range: \(0 \leq h \leq 70\) yd

Exercise 5 [ELL] Students might consider dissecting the graph and verbally explaining each part. Students can write their explanations in their primary language and translate part of their explanations while discussing the problem with their groups.

5. Sample answer: Zeke, the fish, swam slowly, then more rapidly to the bottom of his bowl and stayed there for a while. When Zeke's owner sprinkled fish food into the water, Zeke swam toward the surface to eat. The y-intercept is the fish's depth at the start of the story. The x-intercept represents the time the fish reached the surface of the bowl.

Exercise 6 In each part, students need to decide which variable depends on which. In 6b, distance depends on speed; in 6e, the independent variable is time. Although all of these situations are continuous, it's good for students to ask whether the phenomenon is continuous or discrete. [Alert] The graph of time versus distance would be curved, but this asks for time versus speed. The acceleration due to gravity is constant, so the speed increase is linear.

6a. Time in seconds is the independent variable; the height of the ball in feet is the dependent variable.

6b. The car's speed in miles per hour is the independent variable; the braking distance in feet is the dependent variable.

6c. Time in minutes is the independent variable; the drink's temperature in degrees Fahrenheit is the dependent variable.

Reason and Apply

4. Harold's concentration often wanders from the game of golf to the mathematics involved in his game. His scorecard frequently contains mathematical doodles and graphs.
   a. What is a real-world meaning for this graph found on one of his recent scorecards?
   b. What units might he be using? Possible answer: seconds and yards
   c. Describe a realistic domain and range for this graph.
   d. Does this graph show how far the ball traveled? Explain. No, the horizontal distance traveled is not measured.

5. Make up a story to go with the graph at right. Be sure to interpret the x- and y-intercepts.

6. Sketch what you think is a reasonable graph for each relationship described. In each situation, identify the variables and label your axes appropriately.
   a. the height of a ball during a game of catch with a small child
   b. the distance it takes to brake a car to a full stop, compared to the car's speed when the brakes are first applied
   c. the temperature of an iced drink as it sits on a table for a long period of time
   d. the speed of a falling acorn after a squirrel drops it from the top of an oak tree
   e. your height above the ground as you ride a Ferris wheel

7. Sketch what you think is a reasonable graph for each relationship described. In each situation, identify the variables and label your axes appropriately. In each situation, will the graph be continuous or will it be a collection of discrete points or pieces? Explain why.
   a. the amount of money you have in a savings account that is compounded annually, over a period of several years, assuming no additional deposits are made
   b. the same amount of money that you started with in 7a, hidden under your mattress over the same period of several years
   c. an adult's shoe size compared to the adult's foot length
   d. the price of gasoline at the local station every day for a month
   e. the daily maximum temperature of a town for a month

8. Describe a relationship of your own and draw a graph to go with it. Sample answer: the cost of parking your car at a lot that charges a certain fixed price for up to an hour and then half as much for each additional hour or fraction thereof.

[ELL] In 6e, the term Ferris wheel may be unfamiliar to students; drawing a simple diagram should help students make the link. There is also a picture of a Ferris wheel on page 748.

Exercise 7 [Alert] In 7b, students may want to take inflation into account. The question concerns the amount of money, not its value.
9. Car A and Car B are at the starting line of a race. At the green light, they both accelerate to 60 mi/h in 1 min. The graph at right represents their velocities in relation to time.
   a. Describe the rate of change for each car.
   b. After 1 minute, which car will be in the lead? Explain your reasoning.

Review

4.1 10. Write an equation for the line that fits each situation.
   a. The length of a rope is 1.70 m, and it decreases by 0.12 m for every knot that is tied in it. The length of the rope in meters, and let \( k \) represent the number of knots. \( l = 1.70 - 0.12k \).
   b. When you join a CD club, you get the first 8 CDs for $7.00. After that, your bill increases by $9.50 for each additional CD you purchase. Let \( b \) represent the bill in dollars, and let \( c \) represent the number of CDs purchased; \( b = 7.00 + 9.50(c - 8) \) where \( c \geq 8 \).

3.6 11. APPLICATION Albert starts a business reproducing high-quality copies of pictures. It costs $155 to prepare the picture and then $15 to make each print. Albert plans to sell each print for $27.
   a. Write a cost equation and graph it.
   b. Write an income equation and graph it on the same set of axes.
   c. How many pictures does Albert need to sell before he makes a profit?
   d. What do the graphs tell you about the income and the cost for eight pictures?

1.5 12. APPLICATION Suppose you have a $200,000 home loan with an annual interest rate of 6.5%, compounded monthly.
   a. If you pay $1,200 per month, what balance remains after 20 years? \( y = 142,784.22 \).
   b. If you pay $1,400 per month, what balance remains after 20 years?
   c. If you pay $1,500 per month, what balance remains after 20 years?
   d. Make an observation about the answers to 12a–c. By making an extra $300 payment per month for 20 yr, or $72,000, you save hundreds of thousands of dollars in the long run.

3.7 13. Follow these steps to solve this system of three equations in three variables.
   \[
   \begin{align*}
   2x + 3y - 4z &= -9 \\
   x + 2y + 4z &= 0 \\
   2x - 3y + 2z &= 15
   \end{align*}
   \]
   a. Use the elimination method with Equation 1 and Equation 2 to eliminate \( z \). The result will be an equation in two variables, \( x \) and \( y \).
   b. Use the elimination method with Equation 1 and Equation 3 to eliminate \( z \).
   c. Use your equations from 13a and 13b to solve for \( x \) and \( y \).
   d. Substitute the values from 13c into one of the original equations and solve for \( z \).
   What is the solution to the system?

Exercise 9 The goal of 9a is to relate the slopes of the curves to the rates of change. [Alert] In 9b, students might believe that if the cars reached the same speed in the same amount of time, then the car traveled the same distance. The distance traveled by each car is given by the area of the region between its graph and the horizontal axis.

Exercises 10, 11 Students may note that these are discrete situations. The question is asking for lines that represent the general trends.

Exercise 10b [Alert] Students may be confused about how the equation applies to fewer than 8 CDs. The domain of the function includes only values greater than or equal to 8, although the equation is satisfied by points whose \( x \)-coordinates are less than 8.
Function Notation

Rachel’s parents keep track of her height as she gets older. They plot these values on a graph and connect the points with a smooth curve. For every age you choose on the x-axis, there is only one height that pairs with it on the y-axis. That is, Rachel is only one height at any specific time during her life.

A relation is any relationship between two variables. A function is a special type of relation such that for every value of the independent variable, there is at most one value of the dependent variable. If x is your independent variable, a function pairs at most one y with each x. You can say that Rachel’s height is a function of her age.

You may remember the vertical line test from previous mathematics classes. It helps you determine whether or not a graph represents a function. If no vertical line crosses the graph more than once, then the relation is a function. Take a minute to think about how you could apply this technique to the graph of Rachel’s height and the graph in the next example.

### Function Notation

Function notation emphasizes the dependent relationship between the variables that are used in a function. The notation \( y = f(x) \) indicates that values of the dependent variable, \( y \), are explicitly defined in terms of the independent variable, \( x \), by the function \( f \). You read \( y = f(x) \) as "\( y \) equals \( f \) of \( x \)."

Graphs of functions and relations can be continuous, such as the graph of Rachel’s height, or they can be made up of discrete points, such as a graph of the maximum temperatures for each day of a month. Although real-world data often have an identifiable pattern, a function does not necessarily need to have a rule that connects the two variables.

### Technology Connection

A computer’s desktop represents a function. Each icon, when clicked on, opens only one file, folder, or application.

### Discussing the Lesson

**[Critical Question]** "Does the definition of function require that there be only one value of \( x \) for each value of \( y \)?"  **[Big Idea]** No, the graph need not pass a horizontal line test. You might introduce the term one-to-one to describe a function that has not only one \( y \)-value for every \( x \)-value but also one \( x \)-value for every \( y \)-value. This will be addressed formally in Lesson 5.5.

A function might not be expressible as a rule, either mathematically or verbally. In Chapters 1 and 3, the sequence notation for the \( n \)th term, \( a_n \), can be thought of as a modified function notation. You could replace \( a_n \) with \( u(n) \), which is the way many calculators display the notation.

**[Alert]** Students may think that \( f(x) \) means \( f \) times \( x \) and want to divide by \( x \) or \( f \) to simplify the equation. As needed, point out that \( f(x) \) is an expression in itself and cannot be separated into parts.
**FUNCTIONS**

LESSON EXAMPLE

Using colors when substituting values of \( x \) into the function, as shown in the solution to \( f(8) \), may help students understand the process of evaluating functions.

**[Ask]** “What is happening to the graph when \( x = 3 \)” [Evaluating \( \frac{2x + 5}{x - 3} \) at \( x = 3 \) would require dividing by 0, so the value is undefined.] This observation can lead to a discussion about the domain of \( f(x) \).

**[Alert]** Part e reverses the question and asks for input values instead of output values. This switch may confuse some students.

**Guiding the Investigation**

This is a deepening skills investigation.

This investigation includes several important characteristics of relations and functions. If time is limited, you may want to assign the investigation as homework.

**MODIFYING THE INVESTIGATION**

**Whole Class** Display a–i for the class. Classify each as a function or not, with student input. Discuss students’ reasoning and Step 2.

**Shortened** Skip parts c, f, and i.

**One Step** Pose this problem: “Make a table and a graph of the ages and heights of at least 20 students in this class. Is height a function of age—that is, for every age is there just one height? Is age a function of height?” Encourage students to be creative in measuring ages and heights so that one might be a function of the other. During the discussion, bring out the ideas of the vertical line test and stress that not being one-to-one doesn’t mean that a relation isn’t a function.

Be sensitive to students who might be self-conscious about their height. A measurement is not needed from every student.

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**EXAMPLE**

Function \( f \) is defined by the equation \( f(x) = \frac{2x + 5}{x - 3} \).

Function \( g \) is defined by the graph at right. Find these values.

a. \( f(8) \)

b. \( f(-7) \)

c. \( g(1) \)

d. \( g(-2) \)
e. Find \( x \) when \( g(x) = 0 \).

**Solution**

When a function is defined by an equation, you simply replace each \( x \) with the \( x \)-value and evaluate.

a. \( f(8) = \frac{2(8) + 5}{8 - 3} = \frac{21}{5} = 4.2 \)

b. \( f(-7) = \frac{2(-7) + 5}{-7 - 3} = \frac{-9}{-10} = 0.9 \)

You can check your work with your calculator. See Calculator Note 4A to learn about defining and evaluating functions.

c. The notation \( y = g(x) \) tells you that the values of \( y \) are explicitly defined, in terms of \( x \), by the graph of the function \( g \). To find \( g(1) \), locate the value of \( y \) when \( x = 1 \). The point \((1, 3)\) on the graph means that \( g(1) = 3 \).

d. The point \((-2, 0)\) on the graph means that \( g(-2) = 0 \).

e. To find \( x \) when \( g(x) = 0 \), locate points on the graph with a \( y \)-value of 0. There is only one, at \((-2, 0)\), so \( x = -2 \) when \( g(x) = 0 \).

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**DIFFERENTIATING INSTRUCTION**

**ELL**

Focusing on the vertical line test to determine whether a graph represents a function provides students with a visual connection between the graphs and the definition of function.

**Extra Support**

Give students multiple examples of functional relationships from their own experience (or have them create their own). Have them select \( x \)-values for the scenarios, find the dependent variable, and write equations using \( f(x) \) terminology. This will help students understand the connection between the independent variable and function terminology.

**Advanced**

Have students create their own function and nonfunction graphs and then ask them to find a mathematical model that will produce the graph.

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**LESSON 4.2 Function Notation**

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**TEACHER’S EDITION**

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**DISCOVERING ADVANCED ALGEBRA COURSE SAMPLER**

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In the investigation you will practice identifying functions and using function notation. As you do so, notice how you can identify functions in different forms.

**Investigation**

To Be or Not to Be (a Function)

Below are nine representations of relations.

- **a.** function
- **b.** Not a function; several \( x \)-values are paired with two \( y \)-values each.
- **c.** function
- **d.** function
- **e.** function
- **f.** Not a function; two \( x \)-values are paired with more than one \( y \)-value.
- **g.** independent variable: the age of each student in your class
  - dependent variable: the height of each student
- **h.** independent variable: that automobile’s license plate number
  - dependent variable: the time of sunset

**Step 1g.** Not a function; two students may be the same age but different heights.

**Step 1h.** Function; theoretically every automobile has a unique license plate number.

**Step 1i.** Function; theoretically every automobile has a unique license plate number.

**[Critical Question]** “Does the domain of a relation affect whether it’s a function?” [Big Idea]

The relation could be a function, but it’s not a one-to-one function. To be one-to-one, a function’s graph must pass a horizontal line test.
Step 1 Identify each relation that is also a function. For each relation that is not a function, explain why not.

Step 2 For each graph or table that represents a function in parts a–f, find the y-value when \( x = 2 \), and find the x-value(s) when \( y = 3 \). Write each answer in function notation using the letter of the subpart as the function name. For example, if graph \( a \) represents a function, \( a(2) = \frac{1}{2} \) and \( a(\frac{1}{2}) = 3 \).

Step 2a \( a(2) = 2 \), \( a(0) = 3 \) or \( a(1.5) = 3 \)

Step 2b \( c(2) = 2 \), \( c(1) = 3 \) or \( c(3) = 3 \)

Step 2c \( c(2) \) is undefined, \( c(1) = 3 \) or \( c(3) = 3 \)

Step 2d \( d(2) = 3 \)

When you use function notation to refer to a function, you can use any letter you like. For example, you might use \( y = h(x) \) if the function represents height, or \( y = p(x) \) if the function represents population. Often in describing real-world situations, you use a letter that makes sense. However, to avoid confusion, you should avoid using the independent variable as the function name, as in \( y = x(x) \). Choose freely but choose wisely.

When looking at real-world data, it is often hard to decide whether or not there is a functional relationship. For example, if you measure the height of every student in your class and the weight of his or her backpack, you may collect a data set in which each student height is paired with only one backpack weight. But does that mean no two students of the same height could have backpacks of different weights? Does it mean you shouldn’t try to model the situation with a function? No, two students of the same height could have different backpack weights. You might want to model the data with a function anyway, if a line of fit approximately models the relationship.

### Exercises

#### Practice Your Skills

1. Which of these graphs represent functions? Why or why not?

   - a. Function; each x-value has only one y-value.
   - b. Not a function; there are x-values that are paired with two y-values.
   - c. Function; each x-value has only one y-value.
   - d. Not a function; there are x-values that are paired with two y-values.

2. Use the functions \( f(x) = 3x - 4 \) and \( g(x) = x^2 + 2 \) to find these values.

   - a. \( f(7) \)
   - b. \( g(5) \)
   - c. \( f(-5) \)
   - d. \( g(-3) \)
   - e. \( x \) when \( f(x) = 7 \)

3. Miguel works at an appliance store. He gets paid $7.25 an hour and works 8 hours a day. In addition, he earns a 3% commission on all the items he sells. Let \( x \) represent the total dollar value of the appliances that Miguel sells, and let the function \( m \) represent Miguel’s daily earnings as a function of \( x \). Which function describes how much Miguel earns in a day?

   - A. \( m(x) = 7.25 + 0.03x \)
   - B. \( m(x) = 58 + 0.03x \)
   - C. \( m(x) = 7.25 + 3x \)
   - D. \( m(x) = 58 + 3x \)

#### Support Examples

1. Sketch one graph that represents a function and one that does not represent a function. [Answers will vary.]

2. Use the functions \( f(x) = (x - 2)^2 \) and \( g(x) = -2x + 1 \) to find:

   - a. \( f(-2) \)
   - b. \( g(5) \)
   - c. \( x \) when \( g(x) = 5 \)

#### Closing the Lesson

Reiterate the important points of this lesson: A relation is a relationship between two variables; a function is a relation in which every value of the independent variable corresponds to one and only one value of the dependent variable. If the reverse is also the case, the function is one-to-one. Equivalently, graphs of functions pass the vertical line test. Graphs of one-to-one functions also pass the horizontal line test. Function notation names a function and gives an expression into which other values are substituted to evaluate the function.

[Closing Question] “If \( f(x) = x^2 \), then what are \( f(-3) \) and \( f(5) \)?”

\( f(-3) = 9, f(5) = 25 \)

### Assigning Exercises

- **Suggested Assignments:**
  - **Standard:** 1, 2a, 2e, 4, 5, 7, 8, 13, 16
  - **Enriched:** 4, 5, 7–12, 14, 17

### Exercise Notes

Remind students to explain why for each exercise, even if they’re not asked to.

#### Exercise 1

You might ask students to draw graphs of other nonfunctions. Vertical lines and horizontal parabolas can be included in the extensive variety. In 1c, the dots at the ends of the segments on the graph indicate that the value of the function at that x-value is the negative y-value (corresponding to the filled-in dot) rather than the positive y-value (corresponding to the open dot).

#### Exercise 2d [Alert]

As usual, watch for use of the standard order of operations in squaring the negative number.
Exercise 4 You might want to hand out the Exercise 4 worksheet to prevent students from writing in their books.

Exercise 5 Students could logically argue for opposite choices of the independent variable. For example, in 5d, how far you drive might depend on the amount of gas. Most important is students’ understanding of the process of choosing an independent variable.

5a. The price of the calculator is the independent variable; function.

5b. The time the money has been in the bank is the independent variable; function.

5c. Let \( x \) represent the distance you have driven in miles, and let \( y \) represent the amount of gasoline in your tank in gallons.

5d. Let \( x \) represent the time in months, and let \( y \) represent the account balance in dollars.

5e. Let \( x \) represent the length of your hair.

5f. Let \( x \) represent the time in days, and let \( y \) represent the amount of gasoline in your car’s fuel tank.

4. Use the graph at right to find each value. Each answer will be an integer from 1 to 26. Relate each answer to a letter of the alphabet (1 = A, 2 = B, and so on), and fill in the name of a famous mathematician.

\[
\begin{align*}
\text{a. } f(13) & = R \\
\text{b. } f(25) + f(26) & = E \\
\text{c. } 2f(22) & = N \\
\text{d. } \frac{f(3) + 11}{\sqrt{f(3 + 1)}} & = E \\
\text{e. } \frac{f(1 + 4)}{f(1) + 4} & = \frac{1}{4} \\
\text{f. } x & \text{ when } f(x + 1) = 26 \\
\text{g. } \sqrt{f(21)} + f(14) & = 5 \\
\text{h. } x & \text{ when } 2f(x + 3) = 52 \\
\text{i. } x & \text{ when } f(2x) = 4 \\
\text{j. } f(f(2) + f(3)) & = 18 = R \\
\text{k. } f(9) & = f(25) \\
\text{l. } f(f(5) - f(1)) & = 5 = E \\
\end{align*}
\]

5. Identify the independent variable for each relation. Is the relation a function?

a. the price of a graphing calculator and the sales tax you pay
b. the amount of money in your savings account and the time it has been in the account
c. the amount of gasoline in your car’s fuel tank and how far you have driven since your last fill-up

6. Sketch a reasonable graph for each relation described in Exercise 5. In each situation, identify the variables and label your axes appropriately.

5c. The amount of time since your last haircut is the independent variable; function.

5d. The distance you have driven since your last fill-up is the independent variable; function.

Reason and Apply

7. Suppose \( f(x) = 25 - 0.6x \).

\[
\begin{align*}
\text{a. } & \text{ Draw a graph of this function.} \\
\text{b. } & \text{ What is } f(7)? 20.8 \\
\text{c. } & \text{ Identify the point } (7, f(7)) \text{ by marking it on your graph.} \\
\text{d. } & \text{ Find the value of } x \text{ when } f(x) = 27.4. \text{ Mark this point on your graph.} \\
\end{align*}
\]

8. Identify the domain and range of the function \( g \) in the graph at right. \( \text{domain: } -6 \leq x \leq 5; \text{range: } -2 \leq y \leq 4 \)

Exercise 7 [Ask] “What is a real-world situation that could be represented by this function?” In 7d, students may need to extend their graphs to show the point where \( x \) is negative.

7a, c, d.

Exercise 8 Students may wonder how the graph continues beyond what is drawn. Point out that when a question asks about the domain of a function and only the graph is given, students can assume that the entire graph is showing.
9. Sketch a graph for each function.
   a. \( y = f(x) \) has domain all real numbers and range \( f(x) \leq 0 \).
   b. \( y = g(x) \) has domain \( x > 0 \) and range all real numbers.
   c. \( y = h(x) \) has domain all real numbers and range \( h(x) = 3 \).

10. Consider the function \( f(x) = 3(x + 1)^2 - 4 \).
    a. Find \( f(5) \).
    b. Find \( f(t) \).
    c. Find \( f(x + 2) \).
    d. Use your calculator to graph \( y = f(x) \) and \( y = f(x + 2) \) on the same axes. How do the graphs compare?

11. Kendall walks toward and then away from a motion sensor. Is the \((\text{time, distance})\) graph of his motion a function? Why or why not?

12. APPLICATION The length of a pendulum in inches, \( L \), is a function of its period, or the length of time it takes to swing back and forth, in seconds, \( t \). The function is defined by the formula \( L = 9.73t^2 \). 12a. Approximately 155.68 in.
    a. Find the length of a pendulum if its period is 4 s.
    b. The Foucault pendulum at the Panthéon in Paris has a 62-pound iron ball suspended on a 220-foot wire. What is its period?
    c. Astronomer Jean Bernard Leon Foucault (1819–1868) displayed this pendulum for the first time in 1851. The floor underneath the swinging pendulum was covered in sand, and a pin attached to the ball traced the pendulum’s path. While the ball swung back and forth in straight lines, it changed direction relative to the floor, proving that Earth was rotating underneath it.

13. The number of diagonals of a polygon, \( d \), is a function of the number of sides of the polygon, \( n \), and is given by the formula \( d = \frac{n(n-3)}{2} \).
    a. Find the number of diagonals in a dodecagon (a 12-sided polygon). 54 diagonals.
    b. How many sides would a polygon have if it contained 170 diagonals? 20 sides

Exercise 9 Domains and ranges that are expressed as equations or inequalities can also be expressed in words. For example, the range in 9a is all non-positive numbers, and for 9c, the range is the number 3.

**Exercise 10 [Alert] Students might be confused by 10b and 10c. They need only replace \( x \) with the letter or expression. Expanding or simplifying is unnecessary. In 10d, they can graph on a calculator without squaring \( (x + 2) \). This exercise is a preview of Lessons 4.3 and 4.4.**

10d. The graphs are the same shape. The graph of \( f(x + 2) \) is shifted 2 units to the left of the graph of \( f(x) \).

Exercise 11 If students have not used a motion sensor, you may want to give them a brief explanation. If you do have access to a motion sensor, you should demonstrate it here. Collecting data with a motion sensor is an integral part of many of the investigations in this book.

11. Let \( x \) represent the time since Kendall started moving, and let \( y \) represent his distance from the motion sensor. The graph is a function; Kendall can be at only one position at each moment in time, so there is only one \( y \)-value for each \( x \)-value.

Exercise 12 Students might think that the period is a function of the length rather than the other way around. Either way is legitimate, because the function is one-to-one if the domain is limited to nonnegative values of \( t \). In 12b, the weight of the ball is unneeded information.

Exercise 13b Students might use guess-and-check or a graph if they don’t remember other ways to solve quadratic equations.

**Exercise 13b** Students might use guess-and-check or a graph if they don’t remember other ways to solve quadratic equations.
Exercise 14 [Alert] This exercise might be difficult for students to visualize. You may want to have an interesting bottle and a measuring cup available for students to investigate on their own.

14a.

[Graph showing height vs. time]

14b.

[Graph showing height vs. time]

14c.

[Graph showing height vs. time]

[Advanced] Give students a set of data points (time, height) and have them draw the container that would give those data points.

Review

4.1 14. Create graphs picturing the water height over time as each bottle is filled with water at a constant rate.

a.  

b.  

c.  

2.1 15. APPLICATION The five-number summary of this box plot is $2.10, $4.05, $4.95, $6.80, $11.50. The plot summarizes the amounts of money earned in a recycling fund drive by 32 members of the Oakley High School environmental club. Estimate the total amount of money raised. Explain your reasoning.  

Sample answer: Eight students fall into each quartile. Assuming that the mean of each quartile is the midpoint of the quartile, the total will be 8(3.075 + 4.500 + 5.875 + 9.150), or $180.80.

[Box plot showing money raised]

These photos show the breakdown of a biodegradable plastic during a one-hour period. Created by Australian scientists, the plastic is made of starch and disintegrates rapidly when exposed to water. This technology could help eliminate the 24 million tons of plastic that end up in American landfills every year.

3.6 16. Given the graph at right, find the intersection of lines \( L_1 \) and \( L_2 \).

4.1 17. Sketch a graph for a function that has the following characteristics.

a. domain: \( x \geq 0 \)  
   range: \( f(x) \geq 0 \)  
   linear and increasing

b. domain: \( -10 \leq x \leq 10 \)  
   range: \( -3 \leq f(x) \leq 3 \)  
   nonlinear and increasing

c. domain: \( x \geq 0 \)  
   range: \( -2 < f(x) \leq 10 \)  
   increasing, then decreasing, then increasing, and then decreasing

17a. possible answer:  

17b. possible answer:  

17c. possible answer:  

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0.1 18. You can use rectangle diagrams to represent algebraic expressions. For instance, this diagram demonstrates the equation \((x + 5)(2x + 1) = 2x^2 + 11x + 5\). Fill in the missing values on the edges or in the interior of each rectangle diagram.

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3.6 19. Alice and Carlos are each recording Bao’s distance from where they stand. Initially Bao is between Alice and Carlos, standing 0.2 m from Alice and 4.2 m from Carlos. He walks at 0.5 m/s away from Alice and toward Carlos.

a. On the same axes, sketch graphs of Bao’s distance from each student as a function of time.

b. Write an equation for each graph. \(A(t) = 0.2 + 0.5t\); \(C(t) = 4.2 - 0.5t\)

c. Find the intersection of the graphs and give the real-world meaning of that point. \((4, 2.2)\); After 4 s, Bao is 2.2 m from both Alice and Carlos.

### Project

**STEP FUNCTIONS**

The graph at right represents a step function. The open circles mean that those points are not included in the graph. For example, the value of \(f(3)\) is 5, not 2. The places where the graph “jumps” are called discontinuities.

In Lesson 3.6, Exercise 9, you were introduced to an often-used step function—the greatest integer function, \(f(x) = \lfloor x \rfloor\). Two related functions are the ceiling function, \(f(x) = \lceil x \rceil\), and the floor function, \(f(x) = \lfloor x \rfloor\).

Do further research on the greatest integer function, the ceiling function, and the floor function. Prepare a report or class presentation on the functions. Your project should include:

- A graph of each function.
- A written or verbal description of how each function operates, including any relationships among the three functions. Be sure to explain how you would evaluate each function for different values of \(x\).
- Examples of how each function might be applied in a real-world situation.

As you do your research, you might learn about other step functions that you’d like to include in your project.

### OUTCOMES

- Graphs show the ceiling and floor (greatest integer) functions. The greatest integer function might have its own graph.
- Descriptions are given for each function and for how to evaluate each function for different values of \(x\), including negative values.
- Examples of real-world applications include things such as phone, parking, and postage rates for the ceiling function.
- Other examples of step functions are given, such as the Heaviside step function.

### Supporting the Project

Student web research could start at links from [www.keymath.com](http://www.keymath.com) and include some interesting calculus sites, which might cause students to ask some interesting questions.

### Supporting the Project

**OUTCOMES**

- Graphs show the ceiling and floor (greatest integer) functions. The greatest integer function might have its own graph.
- Descriptions are given for each function and for how to evaluate each function for different values of \(x\), including negative values.
- Examples of real-world applications include things such as phone, parking, and postage rates for the ceiling function.
- Other examples of step functions are given, such as the Heaviside step function.
- The report includes further research on discontinuities.
In Chapter 3, you worked with two forms of linear equations:

- Intercept form: \( y = a + bx \)
- Point-slope form: \( y = y_1 + b(x - x_1) \)

In this lesson you will see how these forms are related to each other graphically.

With the exception of vertical lines, lines are graphs of functions. That means you could write the forms above as \( f(x) = a + bx \) and \( f(x) = f(x_1) + b(x - x_1) \).

The investigation will help you see the effect that moving the graph of a line has on its equation. Moving a graph horizontally or vertically is called a translation. The discoveries you make about translations of lines will also apply to the graphs of other functions.

### Investigation

**Movin’ Around**

In this investigation you will explore what happens to the equation of a linear function when you translate the graph of the line. You’ll then use your discoveries to interpret data. Graph the lines in each step on the same set of axes and look for patterns.

On graph paper, graph the line \( y = 2x \) and then draw a line parallel to it, but 3 units higher. What is the equation of this new line? If \( f(x) = 2x \), what is the equation of the new line in terms of \( f(x) \)?

Draw a line parallel to the line \( y = 2x \), but shifted down 4 units. What is the equation of this line? If \( f(x) = 2x \), what is the equation of the new line in terms of \( f(x) \)?

Mark the point where the line \( y = 2x \) passes through the origin. Plot a point right 3 units from the origin. Draw a line parallel to the original line through this point. Use the point to write an equation in point-slope form for the new line. Then write an equation for the line in terms of \( f(x) \).

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**OBJECTIVES**

- Review linear equations
- Describe translations of a line in terms of horizontal and vertical shifts
- Write the equation of a translated line using \( h \) and \( k \)
- Understand point-slope form as a translation of the line with its equation written in intercept form
- Apply translations to functions
- Apply and identify translations to piecewise-defined functions

**OUTLINE**

One or two days:
- Standard: 35 min 20 min Investigation
- Standard: 10 min 5 min Discuss Investigation
- Standard: 20 min 10 min Examples
- Standard: 25 min 10 min Exercises

**MATERIALS**

- Investigation Worksheet, optional
- motion sensors
- graph paper
- Coordinate Axes (T), optional
- Sketchpad demonstration Lines, optional
- Calculator Note 4C

**ADDITIONAL SUPPORT**

- Lesson 4.3 More Practice Your Skills
- Lesson 4.3 Condensed Lessons (in English or Spanish)
- TestCheck worksheets

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Students who are new to point-slope form or who have no experience with motion sensors will need more time to do this investigation. Although this lesson can be done with pencil and paper, graphing technology will greatly enhance students’ learning. The extra time on Day 2 will allow students who do not have graphing calculators at home to use the school’s calculators.

If necessary, remind students that \( a \) is the \( y \)-intercept, \( b \) is the slope, and \( (x_1, y_1) \) is a point on the line.

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**ONGOING ASSESSMENT**

Observe students’ facility graphing parallel lines, finding equations of lines in point-slope form, and using motion sensors. Also assess their understanding of function notation.

Guiding the Investigation

This is a deepening skills investigation. It is also an activity investigation. You can use the sample data...
The vertex for B is below the vertex for A because B is recorded time when the walker changes direction.

If A’s graph is a translation of B’s graph, about 2 units farther from the walker when the vertex for B because A is recorded time when the walker changes direction.

y = f(x - h) + k

Your group will now use motion sensors to create a function and a translated copy of that function. See Calculator Note 4C for instructions on how to collect and retrieve data from two motion sensors.

Step 9a A’s graph has its vertex farther to the right, indicating A’s recorded time is greater when the walker changes direction.

Step 9b If A’s graph is y = f(x), what equation describes B’s graph? Describe how you determined this equation. y = f(x + 2) - 1, because B is delayed by 2 s and sits about 1 ft closer to C.

c. In general, if the graph of y = f(x) is translated horizontally units and vertically k units, what is the equation of this translated function? y = f(x - h) + k

Guiding the Investigation (continued)

If you do not wish to conduct the investigation as an activity.

MODIFYING THE INVESTIGATION

Whole Class Use graphing calculators to complete Steps 1 through 5 with student input. Have four students demonstrate and complete Steps 6 and 7. Discuss Steps 8 and 9.

Shortened Skip Steps 6 through 9.

One Step Pose this problem in place of the investigation: “What is an equation of the line that results from translating every point on line y = 2x to the right 3 units and up 5 units?” Encourage a variety of approaches. During the discussion, introduce the term translation and encourage the class to look for patterns.

FACILITATING STUDENT WORK

Steps 1, 2 Students may be confused by the phrase “in terms of f(x).” They are to use f(x) in place of x in the equation.

Step 3 As needed, help students see that 2(x - 3) is f(x - 3).

Steps 1–3 As needed, remind students how to find equations of lines given two points and how to find equations of lines parallel to another line.

Steps 7, 8 It would be most efficient and effective to collect one set of data for the class and distribute it among all calculators.

DIFFERENTIATING INSTRUCTION

ELL Do not skip this investigation. Discuss the Language Connection on page 200. Determine whether students can explain the analogy in their own words, in either English or their primary language. Debrief well.

Extra Support At first, students might find the y = y1 + b(x - x1) horizontal translation of y1 to be counterintuitive. If this is the case, take time to have students substitute various values for x1 and observe the effect on the graph.

Advanced Have students examine translations of lines in intercept, point-slope, and standard form so that they can see how the translation differs among the various forms. Have them explain their observations to each other or to the class.

See page 889 for Step 4 and further answer to Step 9.

LESSON 4.3 Lines in Motion 199
Steps 7, 8 (continued)

Having four students act out Step 7 allows the rest of the class to focus on the situation and its mathematical meaning.

ASSESSING PROGRESS

Watch students’ understanding of the idea that the graph of a function can be translated using basic operations on the function expression.

DISCUSSING THE INVESTIGATION

For Step 9c, ask: [Critical Question] “What is the real-world meaning of the translated graph?” [The data collection both began and ended 2 seconds later.]

Ask students to clarify confusion about the vertical height of the graph as representative of the walker’s horizontal distance from the motion detector. The difference in the heights of the graphs represents the horizontal distance between motion detectors. [Ask] “What does the horizontal axis on the graph represent?” [time]

Discussing the Lesson

As you lead the discussion, model the use of the terms map, mapped, and mapping.

As the class focuses on the definition of translation, repeat that if \( h \) is positive, then the translation is to the right, and if \( h \) is negative, then the translation is to the left. Similarly, if \( k \) is positive, then the translation is up, and if \( k \) is negative, then the translation is down. [Critical Question] “How can you remember this?” Students will articulate different ways. One approach is to think of what values of \( x \) and \( y \) give 0 on the left and \( f(0) \) on the right. The origin has shifted to the point \((h, k)\). Another approach is to realize that the equation \( y = f(x - h) + k \) is equivalent to \( y - k = f(x - h) \), so both movement upward and movement to the right involve positive \( h \) and \( k \). [Big Idea] One way to think of translations is as \( (x - h, y - k) \) replacing \( x \) and \( y \), respectively.

Replacing \( x \) with \( (x - h) \) translates the graph \( h \) units horizontally, and replacing \( y \) with \( (y - k) \) translates the graph \( k \) units vertically.

LESSON EXAMPLE A

If students have been working mechanically so far, this example will push them to deeper understanding. You might draw the line on the Coordinate Axes transparency from Chapter 0 and show the movement.

The example shows that for a line, one translation that is horizontal and vertical is also a simple vertical translation. [Critical Question] “Is every translation of a straight line equivalent to a vertical translation?” [Big Idea] It is not for vertical lines. A horizontal translation by \( h \) units of the graph of \( y = a + bx \) gives \( y = a + b(x - h) \), which is equivalent to \( y = (a - bh) + bx \), a vertical translation by the constant \( a - bh \). Students may be skeptical about

If you know the effects of translations, you can write an equation that translates any function on a graph. No matter what the shape of a function \( y = f(x) \) is, the graph of \( y = f(x - 3) + 2 \) will look just the same as \( y = f(x) \), but it will be translated up 2 units and right 3 units. Understanding this relationship will enable you to graph functions and write equations for graphs more easily.

Translation of a Function

A translation moves a graph horizontally or vertically or both.

Given the graph of \( y = f(x) \), the graph of \( y = f(x - h) + k \) or, equivalently, \( y - k = f(x - h) \) is a translation horizontally \( h \) units and vertically \( k \) units.

Language CONNECTION

The word “translation” can refer to the act of converting between two languages. Similar to its usage in mathematics, translation of foreign languages is an attempt to keep meanings parallel. Direct substitution of words often destroys the subtleties of meaning of the original text. The complexity of the art and craft of translation has inspired the formation of Translation Studies programs in universities throughout the world.

In a translation, every point \((x_i, y_i)\) is mapped to a new point, \((x_i + h, y_i + k)\). This new point is called an image of the original point. If you have difficulty remembering which way to move a function, recall the point-slope form of the equation of a line. In \( y = y_i + (bx - x_i) \), the point at \((0, 0)\) is translated to the new point at \((x_i, y_i)\). In fact, every point is translated horizontally \( x \) units and vertically \( y \) units.

EXAMPLE A

Describe how the graph of \( f(x) = 4 + 2(x - 3) \) is a translation of the graph of \( f(x) = 2x \).

The graph of \( f(x) = 4 + 2(x - 3) \) passes through the point \((3, 4)\). Consider this point to be the translated image of \((0, 0)\) on \( f(x) = 2x \).

The point is translated right 3 units and up 4 units from its original location, so the graph of \( f(x) = 4 + 2(x - 3) \) is the graph of \( f(x) = 2x \) translated right 3 units and up 4 units.

Note that you can distribute and combine like terms in \( f(x) = 4 + 2(x - 3) \) to get \( f(x) = -2 + 2x \). The fact that these two equations are equivalent means that translating the graph of \( f(x) = 2x \) right 3 units and up 4 units is equivalent to translating the line down 2 units. In the graph in the example, this appears to be true.
In the investigation and Example A, you translated a line that passed through the origin. If you are translating a graph of a function that does not pass through the origin, then you will need to identify points on the original function that will match up with points on the translated image.

**EXAMPLE B**

The red graph is a translation of the graph of function $f$. Write an equation for the red function in terms of $f(x)$.

> **Solution**

Any point on $f(x)$ can be matched with a point right 2 units and down 3 units on the red function. For example, the image of $(−1, 2)$ is $(1, −1)$. One notation to show this translation is $(x, y) \rightarrow (x+2, y−3)$. The equation of the red function can be written $y = (−3) \cdot f(x−2)$, or $y + 3 = f(x−2)$.

You can describe or graph a transformation of a function graph without knowing the equation of the function. But in the next few lessons, you will find that knowledge of equations for different families of functions can help you learn more about transformations.

**EXERCISES**

### Practice Your Skills

1. The graph of the line $y = \frac{3}{2}x$ is translated right 5 units and down 3 units. Write an equation of the new line.

   $y = −3 + \frac{3}{2}(x−5)$

2. How does the graph of $y = f(x−3)$ compare with the graph of $y = f(x)$? translated right 3 units

3. If $f(x) = −2x$, find

   a. $f(x + 3)$
   b. $−3 + f(x−2)$
   c. $−2(x + 3)$, or $−2x − 6$

   $−3 + (−2)(x−2)$, or $−2x + 1$

   $5 + (−2)(x+1)$, or $−2x + 3$

### SUPPORT EXAMPLES

1. The graph of the line $y = −2x − 1$ is translated $−2$ units horizontally and $6$ units vertically. Write an equation of the new line. $y = −2(x + 2) − 1 + 6$ or $y = −2x + 1$

2. Rewrite $y = f(x)$ as a function that has been translated $−3$ units vertically and $4$ horizontally. $y = f(x−4) − 3$
Exercise 4 Have students find another equation, and show that the two equations are algebraically equivalent.

Exercise 6 [Ask] "Didn't we decide that every horizontal translation is a vertical translation?" [That property holds only for lines.] You might use the Exercise 6 transparency as you discuss this exercise. [Extra Support] If students are confused by these graphs, suggest that they focus on how a single point on the graph moves in order to determine the translation of the entire graph. Then have them verify by checking a second point.

Exercise 7 [Alert] Students may miss the point that the ropes have the same thickness because they're cut from the same source. [Ask] "Why does the rope have to be the same thickness in order to find this equation?" [The equations have the same slope.] "What are the meanings of 102 and 6.3?" [the original length of the rope and the amount it's shortened by each knot]

4. Consider the line that passes through the points (−5.2, 3.18) and (1.4, −4.4), as shown.
   a. Find an equation of the line. @ $y = -4.4 - 1.148(x - 1.4)$ or $y = 3.18 - 1.148(x + 5.2)$
   b. Write an equation of the parallel line that is 2 units above this line.
      $y = -2.8 + 1.148(x + 2)$ or $y = 5.18 - 1.148(x + 5.2)$

5. Write an equation of each line.
   a. the line $y = 4.7x$ translated down 3 units @ $y = -3 + 4.7x$
   b. the line $y = -2.8x$ translated right 2 units $y = -2.8(x - 2)$
   c. the line $y = -x$ translated up 4 units and left 1.5 units $y = 4 - (x + 1.5)$, or $y = 2.5 - x$

Reason and Apply

6. The graph of $y = f(x)$ is shown in black. Write an equation for each of the red image graphs in terms of $f(x)$.
   a. $y = -2 + f(x)$
   b. $y = 2 + f(x - 1)$
   c. $y = -3 + f(x + 2)$
   d. $y = -2 + f(x - 1)$

7. Jeannette and Keegan collect data about the length of a rope as knots are tied in it. The equation that fits their data is $y = 102 - 6.3x$, where $x$ represents the number of knots and $y$ represents the length of the rope in centimeters. Mitch had a piece of rope cut from the same source. Unfortunately he lost his data and can remember only that his rope was 47 cm long after he tied 3 knots. Write an equation that describes Mitch’s rope. $y = 47 - 6.3(x - 3)$
8. Rachel, Pete, and Brian perform Steps 6–9 of the investigation in this lesson. Rachel walks while Pete and Brian hold the motion sensors. A graph of their results is shown at right.

a. The black curve is made from the data collected by Pete’s motion sensor. Where was Brian standing and when did he start his motion sensor to create the red curve?

b. If Pete’s curve is the graph of \( y = f(x) \), what equation represents Brian’s curve? \( y = 1.5 + f(x + 2) \)

9. **Application** Kari’s assignment in her computer programming course is to simulate the motion of an airplane by repeatedly translating it across the screen. The coordinate system in the software program is shown at right. In this program, coordinates to the right and down are positive.

The starting position of the airplane is (1000, 500), and Kari would like the airplane to end at (7000, 4000). She thinks that moving the airplane in 15 equal steps will model the motion well.

a. What should be the airplane’s first position after \((1000, 500)\)? \((1400, 733)\)

b. If the airplane’s position at any time is given by \((x, y)\), what is the next position in terms of \(x\) and \(y\)? \((x + 400, y + 253.3)\)

c. If the plane moves down 175 units and right 300 units in each step, how many steps will it take to reach the final position of \((7000, 4000)\)? 20 steps

**Exercise 8** If students didn’t do the last steps of the investigation, they may need to describe them or do a demonstration at this time.

**Exercise 9** This is a recursive procedure, because each step depends on the previous one.

**Exercise 10** This mini-investigation will take more time than the other exercises, so you might want to assign it to groups. Unlike the coefficients in intercept form or point-slope form, \(a\), \(b\), and \(c\) have no direct interpretation as intercepts or slope.

Because the standard form is not in \( y = \text{form} \) and the coefficient of \(y\) is not necessarily 1, when students just replace \(y\) with \((y - k)\), the constant \(k\) is multiplied by the original coefficient of \(y\).

\[
ax + by = c
\]

Replace \( y \) with \((y - k)\)

\[
ax + by - bk = c
\]

Distribute \(b\)

\[
y = \frac{c - ax + bk}{b}
\]

Subtract \(ax\) and \(by\) from both sides

\[
y = \frac{c - ax + bk}{b}
\]

Simplify

This works the same way for horizontal translations.

It is worth pointing out that when you expand the standard form of the equation for the translated line, the constant is the only coefficient that changes. The \(x\)- and \(y\)-coefficients remain the same as in the equation of the original line.

**[Advanced]** Encourage students to find a way to get the slope from an equation in standard form without having to change the equation to intercept form.
10c. i. y-intercept: 4; slope: $-\frac{4}{3}$  
10c. ii. y-intercept: 5; slope: 1  
10c. iii. y-intercept: $-1$; slope: 7  
10c. iv. y-intercept: $-\frac{1}{2}$; slope: $\frac{1}{2}$  
10c. v. y-intercept: 5; slope: 0  
10c. vi. y-intercept: none; slope: undefined  
10d. ii. $4x + 3y = -8$  
10d. iv. $4x + 3y = 9$  
10d. v. $4x + 3y = 7$  
10d. vi. $4x + 3y = 10$

Exercise 11 [ELI] Students may need some definitions and context with this exercise.

Exercise 13 Encourage variety in solution methods.

**EXTENSIONS**

A. Have students program their calculators to accomplish Kari’s task in Exercise 9.

B. Use Take Another Look activity 1 on page 247.

b. Solve the standard form, $ax + by = c$, for $y$. The result should be an equivalent equation in intercept form. What is the $y$-intercept? What is the slope?  
10b. $y = \frac{5}{4} - \frac{6}{b}; y$-intercept: $\frac{5}{4}$; slope: $-\frac{6}{b}$

c. Use what you’ve learned from 10b to find the $y$-intercept and slope of each of the equations in 10a.

d. The graph of $4x + 3y = 12$ is translated as described below. Write an equation in standard form for each of the translated graphs.

i. a translation right 2 units $4x + 3y = 20$  
ii. a translation left 5 units $4x + 3y = 20$  
iii. a translation up 4 units $4x + 3y = 24$  
iv. a translation down 1 unit $4x + 3y = 24$  
v. a translation right 1 unit and down 3 units $4x + 3y = 24$  
vi. a translation up 2 units and left 2 units $4x + 3y = 24$

e. In general, if the graph of $ax + by = c$ is translated horizontally $h$ and vertically $k$, what is the equation of the translated line in standard form? $ax + by = c + ah + bk$

### Review

3.1 **11. APPLICATION** The Internal Revenue Service has approved ten-year linear depreciation as one method for determining the value of business property. This means that the value declines to zero over a ten-year period, and you can claim a tax exemption in the amount of the value lost each year. Suppose a piece of business equipment costs $12,500 and is depreciated over a ten-year period. At right is a sketch of the linear function that represents this depreciation.

a. What is the $y$-intercept? Give the real-world meaning of this value.

b. What is the $x$-intercept? Give the real-world meaning of this value.

c. What is the slope? Give the real-world meaning of the slope.

d. Write an equation that describes the value of the equipment during the ten-year period. $y = 12,500 - 1,250x$ after 4.8 yr

e. When is the equipment worth $6,500?

2.1 **12.** Suppose that your basketball team’s scores in the first four games of the season were 86 points, 73 points, 76 points, and 90 points.

a. What will be your team’s mean score if the fifth-game score is 79 points? 80.8

b. Write a function that gives the mean score in terms of the fifth-game score. $y = \frac{5}{4}x + 65$

c. What score will give a five-game average of 84 points? 95 points

13. Solve.

a. $2(x + 4) = 38$ $x = 15$  
b. $7 + 0.5(x - 3) = 21$ $x = 31$  
c. $-2 + \frac{3}{4}(x + 1) = -17$ $x = -21$  
d. $4.7 + 2.8(x - 5.1) = 39.7$ $x = 17.6$

3.4 **14.** The three summary points for a data set are $M_1(3, 11)$, $M_2(5, 5)$, and $M_3(9, 2)$. Find the median-median line. $y = \frac{28 - \frac{3}{2}}{2}$
LESSON

4.4

Translations and the Quadratic Family

In the previous lesson, you looked at translations of the graphs of linear functions. Translations can occur in other settings as well. For instance, what will this histogram look like if the teacher decides to add five points to each of the scores? Each bin will shift right 5 units and up 1 unit.

Translations are also a natural feature of the real world, including the world of art. Music can be transposed from one key to another. Melodies are often translated by a certain interval within a composition.

LESSON 4.4 Translations and the Quadratic Family

OBJECTIVES
- Define the parent quadratic function, \( y = x^2 \)
- Determine elements of equations that produce translations of the graphs of parent functions (h and k)
- Introduce the (nonstretched) vertex form of the graph of a parabola, \( y = (x - h)^2 + k \)
- Define parabola, vertex of a parabola, and line of symmetry
- Determine the graph from an equation and the equation from a graph

OUTLINE
One day:
- 20 min Investigation
- 5 min Discuss Investigation
- 5 min Example
- 15 min Exercises

MATERIALS
- Investigation Worksheet, optional
- Two Parabolas (T) for One Step
- Calculator Notes 4G, 4H; 3A, 4D, optional
- Sketchpad demonstration Parabolas, optional
For the exercises:
- geometry software
- Exercise 8 (T), optional
- Calculator Note 4G, optional

ADDITIONAL SUPPORT
- Lesson 4.4 More Practice Your Skills
- Lesson 4.4 Condensed Lessons (in English or Spanish)
- TestCheck worksheets

DIFFERENTIATING INSTRUCTION

ELL
To increase the clarity of the vocabulary surrounding parabolas, such as vertex and line of symmetry, help students create a visual reminder with the specific vocabulary words labeled on the graph. Students could make, present, and display a poster.

Extra Support
Tie this section closely to Lesson 4.3. Reinforce the use of (h, k) to perform translations. If students still struggle, especially with horizontal translation, continue to emphasize viewing the graph, with (h, k) being substituted into the equation.

Advanced
Students can explore extending the idea of translations to cubics and other familiar functions.

Jazz saxophonist Ornette Coleman (b 1930) grew up with strong interests in mathematics and science. Since the 1950s, he has developed award-winning musical theories, such as “free jazz,” which stays from the set standards of harmony and melody.

Music
When a song is in a key that is difficult to sing or play, it can be translated, or transposed, into an easier key. To transpose music means to change the pitch of each note without changing the relationships between the notes.

Translations are also a natural feature of the real world, including the world of art. Music can be transposed from one key to another. Melodies are often translated by a certain interval within a composition.

I see music as the augmentation of a split second of time.
— ERIN CLEARY

When a song is in a key that is difficult to sing or play, it can be translated, or transposed, into an easier key. To transpose music means to change the pitch of each note without changing the relationships between the notes.

Translations are also a natural feature of the real world, including the world of art. Music can be transposed from one key to another. Melodies are often translated by a certain interval within a composition.
This lesson begins a sequence of four lessons that discuss transformations while introducing or reviewing families of relations. Lesson 4.4 extends the discussion of translations to parabolic graphs of quadratic equations. These topics will be explored further in Chapter 7 (Quadratic and Other Polynomial Functions), Chapter 8 (Conic Sections and Rational Functions), and Chapter 13 (Trigonometric Functions). Much of the lesson may be reviewed for students who have used Discovering Geometry or Discovering Algebra.

ONGOING ASSESSMENT
While students investigate, you can begin to see how well they understand the idea of changing a function equation to get a different function with a related graph. Continue to monitor student comfort with function notation and the use of variables in general.

Discussing the Lesson


[Language] The word quadratic comes from the Latin root quadrare, meaning “to square.” The prefix quad is usually used in words like quadrilateral to mean “four”; its use as “two” in quadratic stems from the fact that squared terms were represented as square (four-sided) shapes, as in rectangle diagrams.

Have students graph the equation $y = x^2$ on their calculators. Make a table of $x$- and $y$-values to explore the symmetry of points on either side of the vertex. [Ask] “Where would you place a line of symmetry? What is the equation of that line?”

In mathematics, a change in the size or position of a figure or graph is called a transformation. Translations are one type of transformation. You may recall other types of transformations, such as reflections, dilations, stretches, shrinks, and rotations, from other mathematics classes.

In this lesson you will experiment with translations of the graph of the function $y = x^2$. The special shape of this graph is called a parabola. Parabolas always have a line of symmetry that passes through the parabola’s vertex.

The function $y = x^2$ is a building-block function, or parent function. By transforming the graph of a parent function, you can create infinitely many new functions, or a family of functions. The function $y = x^2$ and all functions created from transformations of its graph are called quadratic functions, because the highest power of $x$ is $x$-squared.

Quadratic functions are very useful, as you will discover throughout this book. You can use functions in the quadratic family to model the height of a projectile as a function of time, or the area of a square as a function of the length of its side.

The focus of this lesson is on writing the quadratic equation of a parabola after a translation and graphing a parabola given its equation. You will see that locating the vertex is fundamental to your success with understanding parabolas.

[Context] Engineering Connection  As the connection on page 408 mentions, a freely hanging cable forms a catenary, not a parabola. When a bridge is hung from cables with its weight evenly distributed, the cables take on a shape close to a parabola.

Several types of bridge designs involve the use of curves modeled by nonlinear functions. Each main cable of a suspension bridge approximates a parabola. To learn more about the design and construction of bridges, see the links at www.keymath.com/DAA.

The Mackinac Bridge in Michigan was built in 1957.
**Investigation**

**Make My Graph**

**Step 1**
Each graph below shows the graph of the parent function $y = x^2$ in black. Find a quadratic equation that produces the congruent, red parabola. Apply what you learned about translations of the graphs of functions in Lesson 4.3.

a. \[ y = x^2 - 4 \]

b. \[ y = x^2 + 1 \]

c. \[ y = (x - 2)^2 \]

d. \[ y = (x + 4)^2 \]

e. \[ y = (x + 2)^2 + 2 \]

f. \[ y = (x - 4)^2 - 2 \]

**Step 2**
Write a few sentences describing any connections you discovered between the graphs of the translated parabolas, the equation for the translated parabola, and the equation of the parent function $y = x^2$.

**Step 3**
In general, what is the equation of the parabola formed when the graph of $y = x^2$ is translated horizontally $h$ units and vertically $k$ units?

\[ y = (x - h)^2 + k \]

The following example shows one simple application involving parabolas and translations of parabolas. In later chapters you will discover many applications of this important mathematical curve.

**FACILITATING STUDENT WORK**

**Shortened**
Discuss Steps 2 and 3 as a class.

**One Step**
Show the Two Parabolas transparency and ask students to experiment on their calculators until they find an equation that produces the parabola drawn with the thicker line. As needed, ask groups whether the methods of translating straight lines in Lesson 4.3 apply to parabolas. During the discussion, formalize the method into a conjecture and ask students to test the conjecture on other examples, such as those from the investigation and the example.
Assessing Progress

Check whether students are making the link between the functions in Lesson 4.3 and the ones in Lesson 4.4. How comfortable are they shifting a parabola in a given direction? Also check to see if they understand how the vertex fits into the new formula.

Discussing the Investigation

[Language] The book uses the term congruent to describe parabolas that are translations of each other. In geometry two polygons are congruent if corresponding sides and corresponding angles are congruent. [Extension] To induce critical thinking, ask, "Is the book correct in using the term congruent?" Encourage discussion that compares and contrasts parabolas and polygons. Unlike a polygon, a parabola has no angles or sides and is not bounded. But a translation of a polygon is indeed congruent; in fact figures can be defined to be congruent if one is the image of the other under translations and rotations.

[Critical Question] "What form of quadratic equations are you using?" [Big idea] All the translations can be represented by the vertex form of a quadratic equation.

In Chapter 7, students will see that this is the vertex form of a quadratic equation, with vertical scale factor \( a = 1. \) [Ask] "What is the line of symmetry of these graphs?" [Alert] Students may have difficulty with the equations of vertical lines.

You might point out that some of the graphs don’t really look parallel and question whether they’re actually translations. Corresponding points of translated parabolas are the same distance apart, but, unlike with lines, the closest points may not be.

Example

This graph shows a portion of a parabola. It represents a diver’s position (horizontal and vertical distance) from the edge of a pool as he dives from a 5 ft long board 25 ft above the water.

a. Identify points on the graph that represent when the diver leaves the board, when he reaches his maximum height, and when he enters the water.

b. Sketch a graph of the diver’s position if he dives from a 10 ft long board 10 ft above the water. (Assume that he leaves the board at the same angle and with the same force.)

c. In the scenario described in part b, what is the diver’s position when he reaches his maximum height?

Solution

a. The point (5, 25) represents the moment when the diver leaves the board, which is 5 ft long and 25 ft high. The vertex, (7.5, 30), represents the position where the diver’s height is at a maximum, or 30 ft; it is also the point where the diver’s motion changes from upward to downward. The \( x \)-intercept, approximately (13.6, 0), indicates that the diver hits the water at approximately 13.6 ft from the edge of the pool.

b. If the length of the board increases from 5 ft to 10 ft, then the parabola translates right 5 units. If the height of the board decreases from 25 ft to 10 ft, then the parabola translates down 15 units. If you define the original parabola as the graph of \( y = f(x) \), then the function for the new graph is \( y = f(x - 5) - 15 \).

c. As with every point on the graph, the vertex translates right 5 units and down 15 units. The new vertex is \( (7.5, 15) \), or (12.5, 15). This means that when the diver’s horizontal distance from the edge of the pool is 12.5 ft, he reaches his maximum height of 15 ft.

You can extend the ideas you’ve learned in translating linear and quadratic functions to functions in general. For a function \( y = f(x) \), to translate the function horizontally \( h \) units, you can replace \( x \) in the equation with \( x - h \). To translate the function vertically \( k \) units, replace \( y \) in the equation with \( y - k \). If you translate the graph of \( y = x^2 \) horizontally \( h \) units and vertically \( k \) units, then the equation of the translated parabola is \( y = (x - h)^2 + k \). You may also see this equation written as \( y + k = (x - h)^2 \) or \( y = k - (x - h)^2 \).

The graphs of all quadratic functions are parabolas. [Advanced] "Is every parabola the graph of a quadratic function?" [If the line of symmetry of the parabola’s graph is vertical, then the parabola is a graph of a function in the family \( y = x^2 \). If the line of symmetry is horizontal, the parabola has the relation \( x = y^2 \) as a parent. Here \( x \) is a quadratic function of \( y \). Rotations of these graphs through a number of degrees other than a multiple of 90° are parabolas in which neither \( x \) nor \( y \) is a function of the other, but they still represent quadratic relations."

You or a student might show the Sketchpad demonstration Transforming Parabolas, or students can use the Dynamic Algebra Exploration at www.keymath.com/DAA to explore these transformations.
It is important to notice that the vertex of the translated parabola is \((h, k)\). That’s why finding the vertex is fundamental to determining translations of parabolas. In every function you study, there will be key points to locate. Finding the relationships between these points and the corresponding points in the parent function enables you to write equations more easily.

2. Describe the location of a parabola \(y = f(x - 3) + 4\) relative to \(y = f(x)\). [translated horizontally 3 and vertically 4]

Closing the Lesson

Reiterate the main point of this lesson: When the graph of the quadratic equation \(y = x^2\) is translated to put its vertex at \((h, k)\), the equation becomes \(y = (x - h)^2 + k\), or, equivalently, \(y - k = (x - h)^2\).

[Closing Question] “Where is the vertex of the parabola represented by \(y = (x + 0)^2 + 4\)” [(0, 4)]

ASSIGNING EXERCISES

Suggested Assignments:
- Standard 1–4, 6, 8, 10, 12
- Enriched 1, 3, 4, 6–9, 11, 14

Types of Exercises:
- Basic 1–5
- Essential 1, 3, 4, 6–9
- Portfolio 9, 16
- Group 7
- Review 12–16

EXERCISE NOTES

Encourage students to describe their reasoning for each exercise, even if the exercise does not directly ask them to do so. In addition to Exercise 16, graphing calculators would be helpful for Exercises 6 and 7 to check answers. However, Exercises 6 and 7 can be done relatively easily without calculator.

Exercise 1 [Extra Support] Remind students to graph their new equations to verify their work.

LESSON EXAMPLE

Whereas the investigation has students translate the graph of the parent function \(y = x^2\), this example asks students to relate two parabolas, neither of which is the parent quadratic function. Students may notice that both graphs actually require a reflection of the graph of \(y = x^2\) across a horizontal line. The example does not require students to write a function for either graph.

[Ask] “Why is the translated vertex \((h, k)\)?” [The vertex of the graph of \(y = x^2\) is \((0, 0)\), so a translation horizontally \(h\) units and vertically \(k\) units puts the translated vertex at \((h, k)\).]

SUPPORT EXAMPLES

1. The parabola \(y = x^2\) is shifted to have a vertex of \((-2, 4)\). What is an equation of this new parabola? \(y = (x + 2)^2 + 4\)
2a. \( y = x^2 - 5 \)

2b. \( y = x^2 + 3 \)

2c. \( y = (x - 3)^2 \)

2d. \( y = (x + 4)^2 \)

2. Each parabola described is congruent to the graph of \( y = x^2 \). Write an equation for each parabola and sketch its graph.
   - a. The parabola is translated vertically \(-5\) units. \( y = (x - 3)^2 - 5 \)
   - b. The parabola is translated vertically \(3\) units. \( y = (x + 4)^2 + 3 \)
   - c. The parabola is translated horizontally \(3\) units. \( y = (x - 2)^2 - 3 \)
   - d. The parabola is translated horizontally \(-4\) units. \( y = (x + 5)^2 - 4 \)

3. If \( f(x) = x^2 \), then the graph of each equation below is a parabola. Describe the location of the parabola relative to the graph of \( f(x) = x^2 \).
   - a. \( y = f(x) - 3 \) translated vertically \(-3\) units
   - b. \( y = f(x) + 4 \) translated vertically \(4\) units
   - c. \( y = f(x - 2) \) translated horizontally \(2\) units
   - d. \( y = f(x + 4) \) translated horizontally \(-4\) units

4. Describe what happens to the graph of \( y = x^2 \) in the following situations.
   - a. \( x \) is replaced with \((x - 3)\). translated horizontally \(3\) units
   - b. \( x \) is replaced with \((x + 3)\). translated horizontally \(-3\) units
   - c. \( y \) is replaced with \((y - 2)\). translated vertically \(-2\) units
d. \( y \) is replaced with \((y + 2)\). translated vertically \(2\) units

5. Solve.
   - a. \( x^2 = 4 \) \( x = 2 \) or \( x = -2 \)
   - b. \( x^2 + 3 = 19 \) \( x = 4 \) or \( x = -4 \)
   - c. \( (x - 2)^2 = 25 \) \( x = 7 \) or \( x = -3 \)

6. Write an equation for each parabola at right.

7. The red parabola below is the image of the graph of \( y = x^2 \) after a horizontal translation of \( 5 \) units and a vertical translation of \(-3\) units.

   a. Write an equation for the red parabola. \( y = (x - 5)^2 - 3 \)
   b. Where is the vertex of the red parabola? \( (5, -3) \)
   c. What are the coordinates of the other four points if they are \( 1 \) or \( 2 \) horizontal units from the vertex? How are the coordinates of each point on the black parabola related to the coordinates of the corresponding point on the red parabola? \( \star \)
   d. What is the length of blue segment \( b \)? Of green segment \( c \)? \( \star \)

8. Given the graph of \( y = f(x) \) at right, draw a graph of each of these related functions.
   - a. \( y = f(x + 2) \)
   - b. \( y = f(x - 1) - 3 \)

Exercise 3 You might ask students to first solve this problem without graphing and then graph to check their answers.

Exercise 5 You can use this exercise to review solving quadratic equations by isolating \( x^2 \) and then taking the square root of both sides. You may want to remind students that nonnegative numbers have two square roots, indicated with the notation \( \pm \), and that the radical symbol alone denotes only the positive square root. Students might find the method of solving by “undoing the order of operations,” from the Chapter 4 Refreshing Your Skills, effective for these exercises. The intersection of the graphs of \( y = x^2 \) and \( y = 4 \) gives the solution of \( x^2 = 4 \).

Exercises 6, 7 [Extra Support] Due to the restrictions of the graphs’ windows, students might mistakenly believe that certain parabolas are “smaller” than others. Emphasize the fact that the parabolas are indeed congruent and that the window limits the full view.

Exercise 7a This exercise uses the fact that pairs of corresponding points are the same distance apart. “Ask” “What is the equation of the line of symmetry?”

Exercise 8 You might use the Exercise 8 transparency as you discuss this problem.
9. **APPLICATION** This table of values compares the number of teams in a pee wee teeball league and the number of games required for each team to play every other team twice (once at home and once away from home). (Review)

<table>
<thead>
<tr>
<th>Number of teams (x)</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of games (y)</td>
<td>0</td>
<td>2</td>
<td>6</td>
</tr>
</tbody>
</table>

a. Continue the table out to 10 teams.
b. Plot each point and describe the graph produced.
c. Write an explicit function for this graph. \( y = (x - 0.5)^2 - 0.25 \)
d. Use your function to find how many games are required if there are 30 teams. 870 games

10. Solve.  
   a. \( 3 + (x - 5)^2 = 19 \)
   b. \( (x + 3)^2 = 49 \)
   c. \( 5 - (x - 1)^2 = -22 \)
   d. \( -5 + (x + 6)^2 = -7 \)

11. This histogram shows the students’ scores on a recent quiz in Ms. Noah’s class. Describe what the histogram will look like if Ms. Noah a. adds five points to everyone’s score.  
    b. subtracts ten points from everyone’s score. 

   The graph will be translated horizontally 5 points (one bin).

### Review

3.1 12. Match each recursive formula with the equation of the line that contains the sequence of points, \((n, u_n)\), generated by the formula.
   a. \( u_n = u_{n-1} + 3 \) where \( n \geq 1 \)  
   b. \( u_n = 3 \)  
   c. \( u_n = u_{n-1} - 8 \) where \( n \geq 2 \)
   A. \( y = 3x - 11 \)
   B. \( y = 3x - 8 \)
   C. \( y = 11 - 8x \)
   D. \( y = -8x + 3 \)

3.6 13. **APPLICATION** You need to rent a car for one day. Mertz Rental charges $32 per day plus $0.10 per mile. Saver Rental charges $24 per day plus $0.18 per mile. Luxury Rental charges $51 per day with unlimited mileage.
   a. Write a cost equation for each rental agency.
   b. Graph the three equations on the same axes.
   c. Describe which rental agency is the cheapest alternative under various circumstances.
   13a. Let \( m \) represent the miles driven, and let \( C \) represent the cost of the one-day rental. Mertz: \( C = 32 + 0.10m \); Saver: \( C = 24 + 0.18m \); Luxury: \( C = 51 \).
   13b. If you plan to drive less than 100 mi, then rent Saver. At exactly 100 mi, Mertz and Saver are the same. If you plan to drive between 100 mi and 190 mi, then rent Mertz. At exactly 190 mi, Mertz and Luxury are the same. If you plan to drive more than 190 mi, then rent Luxury.

### Exercise 9

Because the number of teams and the number of games must be integers, the graph of this function is a collection of points. Its trend can be seen, and predictions made, by drawing a curve through those points.

Students can use differences to find the explicit formula; ask whether it makes sense. For each additional team, you will add double the previous number of teams to represent the new team’s playing each of the existing teams twice.

<table>
<thead>
<tr>
<th>Number of teams (x)</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of games (y)</td>
<td>12</td>
<td>20</td>
<td>30</td>
<td>42</td>
<td>56</td>
<td>72</td>
<td>90</td>
</tr>
</tbody>
</table>

9b. The points appear to be part of a parabola.
Exercise 14 From B to C and from C to E, the graph is actually hyperbolic, but students may not be aware of this subtlety.

Exercise 15 [Alert] Despite the labels on the axes, students may consider the graph as a view from above the walker’s path.

15a. Possible answer: the walker stayed 3.8 m from the sensor for 1.2 s and then walked at a constant 0.84 m/s toward the sensor.

Exercise 16 Students could use geometry software for this investigation. They could create two sliders, a and b, and use those sliders to manipulate the equation \( y = ax + b \). [ELL] Use this mini-investigation to reinforce the effect of varying the values of a and b. It will be beneficial for students to graph an entire family of lines and then create a verbal description of the results.

EXTENSION

Students could create sliders in Fathom or on the TI-Nspire to translate \( y = x^2 \) or other functions.

4.1 14. A car drives at a constant speed along the road pictured at right from point A to point X. Sketch a graph showing the straight line distance between the car and point X as it travels along the road. Mark points A, B, C, D, E, and X on your graph.

4.1 15. The distance between a walker and a stationary observer is shown at right:

a. Describe the actions of the walker.
b. What does the equation \( 3.8 - 0.84(x - 1.2) = 2 \) mean in the context of the graph? When is the walker 2 m from the observer?
c. Solve the equation from 15b and interpret your solution.

16a. The slopes vary, but the y-intercept is always 4.

4.3 16. Use a graphing calculator to investigate the form \( y = ax + b \) of a linear function.

a. On the same coordinate plane, graph the lines \( y = 0.5x + 4, y = x + 4, y = 2x + 4, y = 5x + 4, y = -3x + 4, \) and \( y = -0.25x + 4 \). Describe the graphs of the family of lines \( y = ax + 4 \) as a takes on different values.
b. On the same coordinate plane, graph the lines \( y = 2x - 7, y = 2x - 2, y = 2x, y = 2x + 3, \) and \( y = 2x + 8 \). Describe the graphs of the family of lines \( y = 2x + b \) as b takes on different values. The graphs move up or down, but they all have slope 2.

**IMPROVING YOUR REASONING SKILLS**

**The Dipper**

The group of stars known as the Big Dipper, which is part of the constellation Ursa Major, contains stars at various distances from Earth. Imagine translating the Big Dipper to a new position. Would all of the stars need to be moved the same distance? Why or why not?

Now imagine rotating the Big Dipper around the Earth. Do all the stars need to be moved the same distance? Why or why not?

**IMPROVING REASONING SKILLS**

For the constellation to appear the same from Earth, the stars that are farther away would move a greater distance. If the constellation were translated, all the stars would move the same distance, and it would look different to us. The constellation would look the same to us if it were rotated with Earth as the center of rotation. The stars would move along arcs of great circles on concentric spheres with Earth as the center. The stars farther from Earth would move along arcs with a greater radius and therefore greater length. All the arcs, however, would have the same degree measure.
LESSON 4.5

Reflections and the Square Root Family

The graph of the square root function, \( y = \sqrt{x} \), is another parent function that you can use to illustrate transformations. From the graphs below, what are the domain and range of \( f(x) = \sqrt{x} \)? If you graph \( y = \sqrt{x} \) on your calculator, you can show that \( \sqrt{3} \) is approximately 1.732. What is the approximate value of \( \sqrt{8} \)? How would you use the graph to find \( \sqrt{3} \)? What happens when you try to find \( f(x) \) for values of \( x < 0 \)?

Investigation
Take a Moment to Reflect

In this investigation you first will work with linear functions to discover how to create a new transformation—a reflection. Then you will apply reflections to quadratic functions and square root functions.

Step 1

Graph \( f_1(x) = 0.5x + 2 \) on your calculator.

a. Predict what the graph of \( -f_1(x) \) will look like. Then check your prediction by graphing \( f_1(x) = -f_1(x) \).

b. Change \( f_1 \) to \( f_1(x) = -2x - 4 \), and repeat the instructions in Step 1a.

c. Change \( f_1 \) to \( f_1(x) = x^2 + 1 \) and repeat.

d. In general, how are the graphs of \( y = f(x) \) and \( y = -f(x) \) related?

Graph \( f_2(x) = 0.5x + 2 \) on your calculator.

a. Predict what the graph of \( f_2(x) = -x \) will look like. Then check your prediction by graphing \( f_2(x) = f_2(-x) \).

b. Change \( f_2 \) to \( f_2(x) = -2x - 4 \), and repeat the instructions in Step 2a.

c. Change \( f_2 \) to \( f_2(x) = x^2 + 1 \) and repeat.

Step 2

Explain what happens.

d. Change \( f_2 \) to \( f_2(x) = (x - 3)^2 + 2 \) and repeat.

e. In general, how are the graphs of \( y = f(x) \) and \( y = f(-x) \) related?

\( y = f(-x) \) is a reflection of \( y = f(x) \) across the \( y \)-axis.

DIFFERENTIATING INSTRUCTION

ELL

Use a mirror to discuss the idea of a reflection. Draw a diagram of a person and their reflection and link it to the graph of a function and its reflection using a table of values. Define radical.

Extra Support

Most students recognize vertical reflections over the \( y \)-axis more frequently than they do horizontal reflections over the \( x \)-axis. Be sure to emphasize the difference between \( y = f(-x) \) and \( y = -f(x) \); some students will assume that these are equivalent by using a misinterpretation of the distributive property in function notation. Also, see the ELL note to the left.

Advanced

Piecewise functions open some very interesting avenues of exploration for students. Students can now write equations for some of the very unusual graphs that have been used earlier in this chapter. Explore graphing piecewise functions using Calculator Note 4E.

LESSON 4.5 Reflections and the Square Root Family 213
Teaching the Lesson (continued)

The value of √8 is approximately 2.828. To find √31 on the TI-Nspire, choose Point On from the Points & Lines menu. Select the graph, use the NavPad to point at the segment and press ENTER to set the point on the graph. Students can then grab and drag the point until x = 31 and y = 5.568; you can’t trace to x-values less than 0 because they aren’t in the domain.

ONGOING ASSESSMENT

Assess students’ understanding of both numerical and graphical representations of functions. The investigation is an extension of a common geometry activity; watch to see whether students make that link.

Guiding the Investigation

This is a deepening skills investigation.

Most students can complete this investigation and be prepared to work on the exercises with little or no help from you. For assistance in setting up a good window, see Calculator Note 4D.

MODIFYING THE INVESTIGATION

Whole Class Elicit student predictions for Step 1a. Have students complete Steps 1b–c on calculators. Discuss generalizations. Repeat for Steps 2 and 3.

Shortened Skip Steps 1b, 2d, and 3c.

One Step Ask students to graph the equations y = √x, y = −√x, and write down as many observations about the graphs as they can.

During the discussion, ask about the domains and ranges of these functions and why inserting a negative sign reflects the graph in various ways.

See page 890 for answers to Steps 3a and 3b.

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Facilitating Student Work

Step 1 Calculator Note 4H shows how to use \( f_1 \) in the equation of \( f_c \). As needed, encourage students to do this instead of entering the first equation with the negative sign distributed, possibly forgetting to negate the second term. If students are neglecting to graph equations, suggest that they reread the instructions carefully.

Step 3 Students might struggle with the idea of \( f_1(-x) \) when \( f_1(x) = \sqrt{x} \) thinking they need to take the square root of a negative. Take time to explain that \(-x\) is the opposite of \( x \), so if \( x \) itself is negative, then they will actually be taking the square root of its opposite, which is a positive.

Assessing Progress

Check how well students are developing a mental framework that allows them to see the original function and its properties within the transformed functions.

Reflection of a Function

A reflection is a transformation that flips a graph across a line, creating a mirror image.

Given the graph of \( y = f(x) \),
the graph of \( y = f(-x) \) is a horizontal reflection across the y-axis, and
the graph of \( y = -f(x) \) or \( y = -f(x) \), is a vertical reflection across the x-axis.

Because the graph of the square root function looks like half a parabola, it’s easy to see the effects of reflections. The square root family has many real-world applications, such as dating prehistoric artifacts, as discussed in the Science Connection below.

The next example shows how you can build a piecewise function by choosing particular domains for functions you have previously studied.

Science CONNECTION

Obsidian, a natural volcanic glass, was a popular material for tools and weapons in prehistoric times because it makes a very sharp edge. In 1960, scientists Irving Friedman and Robert L. Smith discovered that obsidian absorbs moisture at a slow, predictable rate and that measuring the thickness of the layer of moisture with a high-power microscope helps determine its age. Therefore, obsidian hydration dating can be used on obsidian artifacts, just as carbon dating can be used on organic remains. The age of prehistoric artifacts is predicted by a square root function similar to \( d = \sqrt{t} \), where \( t \) is time in thousands of years and \( d \) is the thickness of the layer of moisture in microns (millionths of a meter).

These flaked obsidian arrowheads—once used for cutting, carving, and hunting—were made by Native Americans near Jackson Lake, Wyoming, more than 8500 years ago.
**EXAMPLE**

A piecewise function is a function that consists of two or more ordinary functions defined on different domains.

a. Graph \( f(x) = \begin{cases} 2x & -3 \leq x \leq 0 \\ \sqrt{x} & 0 < x \leq 4 \end{cases} \)

b. Find an equation for the piecewise function pictured at right.

**Solution**

a. The graph of the first part is a line with intercept 0 and slope 2. It is defined for \( x \)-values between -3 and 0, so sketch the line but keep only the segment from \((-3, -6)\) to \((0, 0)\).

The second part of the function is a square root function. This part is defined for \( 0 < x \leq 4 \). Graph the function over this domain. [See Calculator Note 4E to learn about graphing piecewise equations on your calculator.]

This completes the graph of \( f(x) \).

b. The graph has two pieces. The left piece appears to be a transformation of the square root parent function \( \sqrt{x} \). The parent function has been reflected horizontally across the \( y \)-axis and translated vertically 1 unit. Starting with \( f(x) = \sqrt{x} \), a horizontal reflection of the function is \( y = f(-x) \), or \( y = \sqrt{-x} \). Translating this function vertically 1 unit, replace \( y \) with \( y - 1 \). This gives the equation \( y - 1 = f(-x) \), or \( y = 1 + \sqrt{-x} \). The domain for this piece of the function is \(-4 \leq x \leq 0 \).

The right piece is a parabola that has been reflected vertically over the \( x \)-axis, and translated horizontally 1 unit and vertically 2 units. Applying these transformations to the parent function \( g(x) = x^2 \) gives the equation \( y - 2 = -g(x - 1) \), or \( y = -(x - 1)^2 + 2 \). The domain is \( 0 < x \leq 3 \). Combining the two pieces, you can represent the piecewise function as

\[
y = \begin{cases} 1 + \sqrt{-x} & -4 \leq x \leq 0 \\ -(x - 1)^2 + 2 & 0 < x \leq 3 \end{cases}
\]

Notice that even though the two pieces meet at \( x = 0 \), you include 0 in only one domain piece. It doesn’t matter which piece, but it should not be included in both.

**DISCUSSING THE INVESTIGATION**

**Language**  Mention that in the notation \( \sqrt{x} \), the symbol \( \sqrt{\phantom{x}} \) is called a radical and the variable \( x \) is called the radicand. Help students become familiar with both terms. For example, \( \sqrt{3} \) can be read as “radical three.” Students may refer to this as “root 3.”

**Critical Question**  Why does the example use the plus or minus sign in front of the radical? If the radical indicates the square root and there are two of them, isn’t the plus or minus sign redundant? [No; the radical refers only to the positive square root.]

**Critical Question**  “How can you remember which variable to replace to make a reflection?” [Big Idea] When we replace \( x \) with \(-x\), values horizontally opposite now act the way the original \( x \)-values did; so the reflection is horizontal, across the \( y \)-axis. Similarly, when \( y \) is replaced, values vertically opposite now act as the original \( y \)-values did so the reflection is across the \( x \)-axis.

As students present their ideas about Step 3c of the investigation, ask **Critical Question**  “What is the range of the function \( f(x) = \sqrt{x} \)?” As students look at the graph, they may conjecture that the range omits some positive numbers, because the graph appears to approach a limit. [Big Idea] Challenging students to find this limit can get them to explore large values of \( x \) and to see that they can get as large a value of \( y \) as they want. You might ask them what \( x \)-value will result in a \( y \)-value of 1000. [\( 1000^2 \), or 1,000,000]

**Extension**  Ask students how they might change a function’s equation to reflect its graph across the line \( y = x \). You need not answer this question now; it foreshadows the exploration on page 220.

**LESSON 4.5 Reflections and the Square Root Family**

As needed, help students realize that there are four parts to the solution, two for each of the two functions. **Extra Support** If the concept of piecewise functions confuses some students, relate the idea to breaking a graph into segments in order to tell a story, as was done in Lesson 4.1. Explain that students will now go beyond telling a story to actually finding a mathematical equation that models the entire graph.

**SUPPORT EXAMPLES**

1. Write an equation for the function \( y = \sqrt{x} \) that has been reflected across the \( y \)-axis and translated up 3. \([y = \sqrt{-x} + 3]\)

2. Describe what happens to the graph \( y = f(x) \) when it is transformed into \( y = -f(-x) + 2 \). [reflected across \( x \)-axis, reflected over \( y \)-axis, translated up 2]
Closing the Lesson

The major points of this lesson are that the graph of \( y = -f(x) \) is a reflection of the graph of \( f(x) \) across the \( x \)-axis and that the graph of \( y = f(-x) \) is a reflection of the same graph across the \( y \)-axis. The lesson also introduces the square root function, \( f(x) = \sqrt{x} \), whose domain and range are the nonnegative real numbers.

(Closing Question) “What equation represents a reflection of the graph \( y = \sqrt{x} \) across both axes?” 
\[ y = -\sqrt{-x} \] or \( y = -\sqrt{x} \)

“What equation represents a reflection of the graph of \( y = x \) across both axes?” \( y = x \)

Assigning Exercises

Suggested Assignments:
- Standard 1, 2, 4, 5, 6, 9, 11, 12, 15
- Enriched 3–8, 10, 11, 14

Types of Exercises:
- Basic 1–5
- Essential 4, 5, 6, 11
- Portfolio 13
- Group 13, 14
- Review 14–19

Exercise Notes

The exercises include practice with all the parent functions and transformations learned to this point. In addition to Exercises 7 and 13, graphing calculators would be helpful for Exercises 1, 2, 4, 6, 9, 11, 12, 15, and 16 to check answers, but these exercises can be done relatively easily without a calculator.

Exercise 2 As needed, suggest that students graph the equations.

Exercise 3, 4 You might use the transparencies for Exercises 3 and 4 as you discuss the exercises.

(Extra Support) If students are having difficulty reflecting the entire graph at once, encourage them to reflect each of the four marked points separately before reconnecting the segments. Students could use tracing paper to trace the function \( y = f(x) \) and perform the reflection in one step.

Exercises

Practice Your Skills

1. Each graph at right is a transformation of the graph of the parent function \( y = \sqrt{x} \). Write an equation for each graph.

2. Describe what happens to the graph of \( y = \sqrt{x} \) in the following situations.
   a. \( x \) is replaced with \( (x - 3) \)
   b. \( x \) is replaced with \( (x + 3) \)
   c. \( y \) is replaced with \( (y - 2) \)
   d. \( y \) is replaced with \( (y + 2) \)

3. Each graph at right is a transformation of the piecewise function \( f(x) \).
   Match each equation to a graph.
   a. \( y = f(-x) \)
   b. \( y = -f(x) \)
   c. \( y = -f(-x) \)

4. Given the graph of \( y = f(x) \) below, draw a graph of each of these related functions.
   a. \( y = f(-x) \)
   b. \( y = -f(x) \)
   c. \( y = -f(-x) \)

5. Each curve at right is a transformation of the graph of the parent function \( y = \sqrt{x} \). Write an equation for each curve.
   a. \( y = \sqrt{-x} \)
   b. \( y = \sqrt{(x - 2)} - 3 \)
   c. \( y = \sqrt{-x} + 2 - 3 \)
   d. \( y = -\sqrt{x} \)
   e. \( y = \sqrt{-x - 2} + 3 \)

You’ve seen in previous lessons that you can transform complicated graphs without knowing their equations. However, writing the equations of piecewise graphs can give you practice working with transformations of the families of graphs you are studying in this chapter, as well as more practice working with domain and range.

Exercises

Practice Your Skills

1. \( y = \sqrt{x} + 3 \)
2. \( y = \sqrt{x} + 3 \)
3. \( y = \sqrt{x} - 3 + 1 \)
4. \( y = \sqrt{x} - 1 - 4 \)

1a. \( y = \sqrt{x} + 3 \)
1b. \( y = \sqrt{x} + 3 \)
1d. \( y = \sqrt{x} - 3 + 1 \)
1e. \( y = \sqrt{x} - 1 - 4 \)

You will need

A graphing calculator for Exercises 7 and 13.
6. Consider the parent function \( f(x) = \sqrt{x} \).

   a. Name three pairs of integer coordinates that are on the graph of \( y = f(x + 4) - 2 \).

   b. Write \( y = f(x + 4) - 2 \) using a radical, or square root symbol, and graph it.

   c. Write \( y = -f(x - 2) + 3 \) using a radical, and graph it.

7. Consider the parabola at right.

   a. Graph the parabola on your calculator. What two functions did you use?

   b. Combine both functions from 7a using ± notation to create a single relation. Square both sides of the relation. What is the resulting equation?

   c. Square both sides of each equation in 8b. What is the resulting equation of each parabola?

   8. Refer to the two parabolas at right.

   a. Explain why neither graph represents a function.

   b. Write a single equation for each parabola using ± notation. If possible answers: \((x, y) = \pm \sqrt{x} + 2\).

   c. Square both sides of each equation in 8b. What is the resulting equation of each parabola?

9. As Jake and Arthur travel together from Detroit to Chicago, each makes a graph relating time and distance. Jake, who lives in Detroit and keeps his watch on Detroit time, graphs his distance from Detroit. Arthur, who lives in Chicago and keeps his watch on Chicago time (1 hour earlier than Detroit), graphs his distance from Chicago.

   a. Sketch what you think each graph might look like.

   b. If Jake’s graph is described by the function \( y = f(x) \), what function describes Arthur’s graph? \( y = -f(x + 1) + 250 \)

   c. If Arthur’s graph is described by the function \( y = g(x) \), what function describes Jake’s graph? \( y = g(x - 1) + 250 \)

10. Write the equation of each parabola. Each parabola is a transformation of the graph of the parent function \( y = x^2 \).

11. Write the equation of a parabola that is congruent to the graph of \( y = -(x + 3)^2 + 4 \), but translated right 5 units and down 2 units.

Exercise 6 [Alert] In 6b and 6c, students may enclose the entire right side of the equation under the radical. Suggest that they graph \( f_1 = \sqrt{x}, f_2 = f(x + 4) - 2 \), and \( f_3 \) (the equation they wrote) to see whether the graphs of \( f_2 \) and \( f_3 \) agree.
Exercise 12 Students need not graph piecewise function \( g(x) \) to complete this exercise, though they should if your standards warrant it. If they choose to, they may be confused by the fact that the function is not continuous. If students aren’t sure how to find \( g(2) \), you might [Ask] “Which piece of the function contains 2 as part of its domain?” \([g(x) = 3]\) [ELL] To help students better understand the evaluation of piecewise functions, have them sketch the graph first. As students use the graph to find the outputs for the respective inputs, relate this skill to finding where the given input falls in the different domains of the definition and then using the respective rule to verify the output.

Exercise 13a [Alert] Students may not understand that they’re being asked simply to substitute 0.7 for \( f \).

13b.

\[ S = f(D) = \frac{1}{63}(5.5)^2 \]

13d. \( D = \frac{1}{63}(5.5)^2 \); the minimum braking distance, when the speed is known

13e.

\[ t = \frac{1}{2}x^2 \]

It is a parabola, but the negative half is not used because the distance cannot be negative.

Exercise 14 Encourage critical thinking to establish in students the tendency to doubt that expressions are functions. See Lesson 4.2 for review. [Alert] In 14a, students may not know that a state may have more than one area code.

12. Let \( f(x) \) be defined as the piecewise function graphed at right, and let \( g(x) \) be defined as

\[
g(x) = \begin{cases} 3 & 0 \leq x \leq 2 \\ 2 + 0.5(x - 2) & 2 < x \leq 4 \\ 2 - (x - 4) & 4 < x \leq 6 \\ 1 & 6 < x \leq 7 \end{cases}
\]

Find each value.

a. \( f(0) \)

b. \( x \) when \( f(x) = 0 \)

c. \( x \) when \( f(x) = 1 \)

d. \( g(1.8) \)

e. \( g(2) \)

f. \( g(4) \)

g. \( g(6.999) \)

13. APPLICATION Police measure the lengths of skid marks to determine the initial speed of a vehicle before the brakes were applied. Many variables, such as the type of road surface and weather conditions, play an important role in determining the speed. The formula used to determine the initial speed is \( S = 5.5\sqrt{D \cdot f} \), where \( S \) is the speed in miles per hour, \( D \) is the average length of the skid marks in feet, and \( f \) is a constant called the “drag factor.” At a particular accident scene, assume it is known that the road surface has a drag factor of 0.7.

a. Write an equation that will determine the initial speed on this road as a function of the lengths of skid marks. \( S = 5.5\sqrt{D \cdot f} \)

b. Sketch a graph of this function.

c. If the average length of the skid marks is 60 feet, estimate the initial speed of the car when the brakes were applied. approximately 36 mi/h

d. Solve your equation from 13a for \( D \). What can you determine using this equation?

e. Graph your equation from 13d. What shape is it?

f. If you traveled on this road at a speed of 65 miles per hour and suddenly slammed on your brakes, how long would your skid marks be?

approximately 199.5 ft

Review

4.2 14. Identify each relation that is also a function. For each relation that is not a function, explain why not.

a. independent variable: state

b. independent variable: any pair of whole numbers

c. independent variable: any pair of fractions

d. independent variable: the day of the year

e. Not a function; many states have more than one area code.

f. Not a function; there are many common denominators for any pair of fractions.
15. Solve for $x$. Solving square root equations often results in extraneous solutions, or answers that don’t work in the original equation, so be sure to check your work.

a. $3 + \sqrt{x - 4} = 20$ \hspace{1cm} x = 293  

b. $\sqrt{2x + 7} = -3$ \hspace{1cm} no solution  
c. $4 - (x - 2)^2 = -21$ \hspace{1cm} $x = 7$ or $x = -3$  
d. $5 - \sqrt{-x + 4} = 2$ \hspace{1cm} $x = -13$

16. Find the equation of the parabola with vertex $(−6, 4)$, a vertical line of symmetry, and containing the point $(−5, 5)$. \hspace{1cm} $y = (x + 6)^2 + 4$

17. The graph of the line $\ell_1$ is shown at right.

a. Write the equation of the line $\ell_1$. \hspace{1cm} $y = \frac{1}{2}x + 5$

b. The line $\ell_1$ is the image of the line $\ell_2$ translated right 8 units. Sketch the line $\ell_2$, and write its equation in a way that shows the horizontal translation.

c. The line $\ell_2$ also can be thought of as the image of the line $\ell_1$ after a vertical translation. Write the equation of the line $\ell_2$ in a way that shows the vertical translation. \hspace{1cm} $y = \frac{1}{2}x + 4$

d. Show that the equations in 17b and c are equivalent. Both equations are equivalent to $y = \frac{1}{2}x + 1$.

18. Consider this data set: $\{37, 40, 36, 37, 39, 47, 40, 38, 35, 46, 43, 40, 47, 49, 70, 65, 50, 73\}$

a. Give the five-number summary. \hspace{1cm} $35, 37.5, 41.5, 49, 73$

b. Display the data in a box plot.

c. Find the interquartile range. \hspace{1cm} 11.5

d. Identify any outliers, based on the interquartile range. \hspace{1cm} 70 and 73

19. Find the intersection of the lines $2x + y = 23$ and $3x − y = 17$. \hspace{1cm} $(8, 7)$

Improving Your Geometry Skills

**Lines in Motion Revisited**

Imagine that a line is translated in a direction perpendicular to it, creating a parallel line. What vertical and horizontal translations would be equivalent to the translation along the perpendicular path? Find the slope of each line pictured. How does the ratio of the translations compare to the slope of the lines? Find answers both for the specific lines shown and, more generally, for any pair of parallel lines.

**Extending Your Geometry Skills**

In general, any translation of a line has the same slope as the original, so the translation amounts don’t relate to the line’s slope. If each point on the line is translated by the same amounts to a point on the perpendicular to the line at that point, however, then there is a relationship. In the example, the vertical translation is $-4$, and the horizontal translation is 3. The ratio $-\frac{4}{3}$ is the slope of the perpendicular line, by the definition of slope. The fact that the slope of the original line is $\frac{1}{3}$ gives a clue about the general case: If every point on a line is translated along a perpendicular line horizontally $a$ units and vertically $b$ units, then $-\frac{b}{a}$ is the slope of the line.
EXPLORATION

OBJECTIVES
- Explore compositions of transformations
- Understand rotation as a composition of two reflections

OUTLINE
One day:
30 min Activity
15 min Discuss Activity

MATERIALS
- The Geometer’s Sketchpad

TEACHING THE EXPLORATION

This will be a review for students who studied rotations as compositions of reflections in Discovering Geometry.

The Dynamic Algebra Exploration at www.keymath.com/DAA can help students visualize the transformations.

ONGOING ASSESSMENT

You can assess how students have internalized the vocabulary of the transformations they have encountered so far.

Guiding the Activity

Step 3
To check the results of transformations, students can select corresponding vertices of the original and the rotated images and choose Coordinates from the Measure menu. Challenge students to find one set of transformations that involves only rotations.

QUESTION NOTES

Question 2 [Ask] “What reflection negates the x-coordinate?” [reflection across the y-axis] “What reflection exchanges coordinates?” [reflection across the line y = x]
Dilations and the Absolute-Value Family

Hao and Dayita ride the subway to school each day. They live on the same east-west subway route. Hao lives 7.4 miles west of the school, and Dayita lives 5.2 miles east of the school. This information is shown on the number line below.

The distance between two points is always positive. However, if you calculate Hao’s distance from school, or HS, by subtracting his starting position from his ending position, you get a negative value:

\[-7.4 - 0 = -7.4\]

In order to make the distance positive, you use the absolute-value function, which gives the magnitude of a number, or its distance from zero on a number line. For example, the absolute value of -3 is 3, or \(|-3| = 3\). For Hao’s distance from school, you use the absolute-value function to calculate

\[HS = |-7.4 - 0| = |-7.4| = 7.4\]

What is the distance from D to HP? What is the distance from H to DP? 12.6; 12.6

In this lesson you will explore transformations of the graph of the parent function \(y = |x|\). See Calculator Note 4F to learn how to graph the absolute-value function. You will write and use equations in the form \(\frac{x}{b} \pm \frac{|y|}{a}\). What you have learned about translating and reflecting other graphs will apply to these functions as well. You will also learn about transformations called dilations that stretch and shrink a graph.

You may have learned about dilations of geometric figures in an earlier course. Now you will apply dilations to functions.

If you dilate a figure by the same scale factor both vertically and horizontally, then the image and the original figure will be similar and perhaps congruent.

If you dilate by different vertical and horizontal scale factors, then the image and the original figure will not be similar.

**LESSON 4.6 Dilations and the Absolute-Value Family**

**OBJECTIVES**
- Define absolute value and its notation and use it to model distance
- Define the parent absolute-value function, \(y = |x|\), and the absolute-value family, \(y = \frac{x}{a} - \frac{b}{k}\)
- Calculate horizontal and vertical scale factors from points on the image of a graph
- Apply horizontal and vertical dilations to functions in general

**OUTLINE**

Two days:
First day:
- 30 min Examples
- 15 min Exercises
Second day:
- 25 min Investigation
- 5 min Discuss Investigation
- 15 min Exercises

**MATERIALS**
- Investigation Worksheet, optional
- string
- small weights
- stopwatches, or watches with seconds hand
- metersticks or tape measures
- graph paper, optional
- Find My Equation (W) for One Step
- Sketchpad demonstration Absolute Value, optional
- Calculator Note 4F, 4G

For the exercises:
- Fathom demonstration Science Fair, optional

**ADDITIONAL SUPPORT**
- Lesson 4.6 More Practice Your Skills
- Lesson 4.6 Condensed Lessons (in English or Spanish)
- TestCheck worksheets
TEACHING THE LESSON

This is the third lesson in the sequence discussing transformations. Here the book focuses on dilations, with examples primarily from absolute-value and square root functions. Much of the lesson may be review for students who have used Discovering Algebra.

You could turn Example B into an investigation by having students collect their own data using a motion sensor and a ball. You would follow the procedure for data collection in the Lesson 1.2 investigation. Use the term dilation rather than stretch or shrink whenever possible. It’s clearer, for example, to say “a vertical dilation by a factor of $\frac{1}{2}$” than “a vertical stretch by a factor of $\frac{1}{2}$,” which actually is a shrink.

ONGOING ASSESSMENT

Assess students’ familiarity with the absolute-value and square root functions and see how well they understand the transformations from the chapter so far.

Discussing the Lesson

The absolute-value function models distance. Distances between the homes on page 221: $DH = |5.2 - (-7.4)| = 12.6$ and $HD = |-7.4 - 5.2| = |12.6| = 12.6$.

[Context] The French mathematician Augustin-Louis Cauchy (1789–1857) first described the absolute-value function in the 1820s. In 1841, the German mathematician Karl Weierstrass (1815–1897) introduced the absolute-value symbol used today.

LESSON EXAMPLE A

The vertical dilation in part a can also be thought of as a horizontal dilation, because $y = 2|x|$ is the same as $y = |2x|$ or $y = \frac{1}{2}x$. Similarly, the horizontal dilation of part b is a vertical dilation, and the combination of part c is equivalent to either a vertical dilation ($y = \frac{3}{2}|x|$) or a horizontal dilation ($y = \frac{1}{2}|x|$).

LESSON EXAMPLE B

This example is an important illustration of a composition of translations, dilations, and reflections of the quadratic family of equations in a real-world context. Students might appreciate seeing a more gradual solution. Graph the translation, $f(x) = (x - 0.86)^2 + 0.6$. Graph the reflection across the vertical line $y = 0.6$, $f(x) = -(x - 0.86)^2 + 0.6$. Pick a data point, such as $(1.14, 0.18)$. Because this point is $1.14 - 0.86$, or $0.28$, unit to the right of the vertex, if the graph were simply a translation of the graph of $y = -x^2$, then the $y$-coordinate would be $0.28^2$, or $0.078$, unit lower than the vertex. But $0.18$ is $0.42$ unit.

EXAMPLE A

Graph the function $y = |x|$ with each of these functions. How does the graph of each function compare to the original graph?

a. $\frac{y}{2} = |x|$ (solution)

b. $y = \frac{x}{3}$ (solution)

c. $\frac{y}{2} = \frac{x}{3}$

Solution

In the graph of each function, the vertex remains at the origin. Notice, however, how the points $(1, 1)$ and $(-2, 2)$ on the parent function are mapped to a new location.

a. Replacing $y$ with $\frac{y}{2}$ pairs each $x$-value with twice the corresponding $y$-value in the parent function. The graph of $\frac{y}{2} = |x|$ is a vertical stretch, or a vertical dilation, of the graph of $y = |x|$ by a factor of 2.

b. Replacing $x$ with $\frac{x}{3}$ multiplies the $x$-coordinates by a factor of 3. The graph of $y = \frac{x}{3}$ is an horizontal stretch, or a horizontal dilation, of the graph of $y = |x|$ by a factor of 3.

c. The combination of replacing $y$ with $\frac{y}{2}$ and replacing $x$ with $\frac{x}{3}$ results in a vertical dilation by a factor of 2 and a horizontal dilation by a factor of 3.

Translations and reflections are rigid transformations—they produce an image that is congruent to the original figure. Vertical and horizontal dilations are nonrigid transformations—the image is not congruent to the original figure (unless you use a factor of 1 or $-1$).

Using what you know about translations, reflections, and dilations, you can fit functions to data by locating only a few key points. For quadratic, square root, and absolute-value functions, first locate the vertex of the graph. Then use any other point to find the factors by which to dilate the image horizontally and/or vertically.
EXAMPLE B

These data are from one bounce of a ball. Find an equation that fits the data over this domain.

<table>
<thead>
<tr>
<th>Time (s)</th>
<th>Height (m)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.54</td>
<td>0.05</td>
</tr>
<tr>
<td>0.58</td>
<td>0.18</td>
</tr>
<tr>
<td>0.62</td>
<td>0.29</td>
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<tr>
<td>0.66</td>
<td>0.39</td>
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<td>0.70</td>
<td>0.46</td>
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<tr>
<td>0.74</td>
<td>0.52</td>
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<td>0.82</td>
<td>0.59</td>
</tr>
<tr>
<td>0.86</td>
<td>0.60</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Time (s)</th>
<th>Height (m)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.90</td>
<td>0.59</td>
</tr>
<tr>
<td>0.94</td>
<td>0.57</td>
</tr>
<tr>
<td>0.98</td>
<td>0.32</td>
</tr>
<tr>
<td>1.02</td>
<td>0.46</td>
</tr>
<tr>
<td>1.06</td>
<td>0.39</td>
</tr>
<tr>
<td>1.10</td>
<td>0.29</td>
</tr>
<tr>
<td>1.14</td>
<td>0.18</td>
</tr>
<tr>
<td>1.18</td>
<td>0.05</td>
</tr>
</tbody>
</table>

The graph appears to be a parabola. However, the parent function \( y = x^2 \) has been reflected, translated, and dilated. Start by determining the translations. The vertex has been translated from \((0, 0)\) to \((0.86, 0.60)\). This is enough information for you to write the equation in the form

\[ y = (x - h)^2 + k \]

for \((x, y)\) and \((h, k)\) the vertex. You could also write the equivalent equation,

\[ y = (x - 0.86)^2 + 0.6. \]

The graph still needs to be reflected and dilated. Select one other data point to determine the horizontal and vertical scale factors. You can use any point, but you will get a better fit if you choose one that is not too close to the vertex. For example, you can choose the data point \((1.14, 0.18)\).

Assume this data point is the image of the point \((1, 1)\) in the parent parabola \( y = x^2 \). In the graph of \( y = x^2 \), \((1, 1)\) is 1 unit away from the vertex \((0, 0)\) both horizontally and vertically. The data point we chose in this graph, \((1.14, 0.18)\), is \(0.14 = 0.86\), or 0.28, unit away from the \(x\)-coordinate of the vertex, and \(0.18 = 0.60\), or \(-0.42\), unit away from the \(y\)-coordinate of the vertex.

\( f(x) = x^2 \)

lower than the vertex, so the graph is dilated vertically by a factor of \(\frac{0.32}{0.28} = \frac{0.32}{0.28} \) or approximately 3.56. Indeed, a graph of \( f(x) = -5.36(x - 0.86)^2 + 0.6\) passes very close to all data points, and the equation is equivalent to the solution given in the book.

Students can use Calculator Note 4G to transform many functions on the calculator. Discovering Algebra Calculator Note 8D presents a calculator program, PARAB, that gives the graph of a parabola and challenges students to write its equation. The program allows students to compare their equation to the original by looking at either a graph or a table of values. Students can use the program to practice problems similar to Example B. You can access this calculator note and the program at www.keymath.com/DA.

In discussing Example B, [Ask] “How can the book assume that the data point \((1.14, 0.18)\) is the image of \((1, 1)\)? What if some other point on the new curve is the image of \((1, 1)\)? For example, what if we assume that data point \((0.54, 0.05)\) is the image of \((1, 1)\)?” [This data point is 0.54 – 0.86, or \(-0.32\), from the vertex horizontally and 0.05 – 0.60, or \(-0.55\), from the vertex vertically, so the new equation is \( y = -0.55 \) \((x - 0.86)^2\). This equation is equivalent to \( y = -5.36(x - 0.86)^2 + 0.6\), very close to the equation in the example, which can be rewritten as \( y = -5.36(x - 0.86)^2 + 0.6\). Students might see that the equations are close because \(-0.42 \approx -0.55\). In general, for a parabola, \( \frac{b}{a} \) is constant (where \( b \) and \( a \) are the vertical and horizontal scale factors, respectively).

[Ask] “How would you generalize this for any function?” In general, a vertical dilation of \( y = f(x) \) by a factor of \( b \) gives \( y_b = bx \). A horizontal dilation of that function by a factor of \( a \) gives \( y_f = f(x - a) \). A horizontal translation of \( h \) and a vertical translation of \( k \) gives \( \frac{(y - k)}{b} = f(x - a) \). [Critical Question] “Why do you divide rather than multiply a variable by the scale factor to change an equation?” [Big Idea] When a variable is divided by a constant, the divided value plays the same role in the equation that the original variable did. So if, for example, \( x \) is divided by 3, values of \( x \) that are 3 times as large will now have the same effect on the equation that the original values of \( x \) had. Wonder aloud whether lines not through the origin also can be thought of as transformations of the parent line, \( y = x \). A vertical translation of the same line by an amount \( a \) gives the familiar equation \( y = a + bx \), so every nonvertical line is a dilation followed by a translation of the parent line, \( y = x \).
Guiding the Investigation

This is a deepening skills investigation. It is also an activity investigation. You can use the sample data if you do not wish to conduct the investigation as an activity.

MODIFYING THE INVESTIGATION

Whole Class Have three students collect data for the whole class. Students can then do Step 2 and discuss Step 3. Have students do Step 4 and discuss Step 5 as a class.

Shortened Use the sample data.

One Step Hand out the Find My Equation worksheet (or display the graphs on an overhead calculator) and ask students to find at least two equations for each mystery graph. As needed, remind them of the meaning of the absolute-value function. As groups finish their work, ask them to create (on graph paper) mystery graphs involving transformations of the graph of the square root function and to exchange them with each other as challenges. During the discussion, formalize the rules for dilations and review the rules for translations and reflections. The equations for the graphs on the worksheet:

a. \( y = \frac{1}{2} |x| \)
b. \( y = 2|x| \)
c. \( y = \frac{1}{4}(x - 1)^2 \)
d. \( y = 3(x - 1)^2 \)
e. \( y = \frac{1}{2}f(x - 2) \)
f. \( y = \frac{1}{5}f(x - 1) \)

FACILITATING STUDENT WORK

Give students a string at least 2 m long. Encourage students to use a variety of string lengths, including several very short lengths and at least one very long length. If they don’t cut their string, they can collect more data later.

Students may wonder whether the measure of the arc of the swing or the amount of weight will affect the period. (Encourage students to test these parameters if there is time. As long as the horizontal displacement of the weight is small compared to the length of the pendulum, the angle measure does not affect the period.)

Theoretically, the period of a pendulum swinging without resistance is given by \( 2\pi \sqrt{\frac{L}{g}} \), where \( L \) is the length and \( g \) is the gravitational constant. If students measure in centimeters, \( g \) is about 980 cm/s\(^2\), so they’ll get about 0.2\sqrt{\frac{L}{L}}. If they measure in inches, \( g \) is about 384 in./s\(^2\), so they’ll get about 0.32\sqrt{\frac{L}{L}}.

For some simple functions, assuming that one point is the image of another can determine the two dilation factors for the function. Using a different pair of points gives the same graph and equivalent equations. For more complicated functions and relations, students must check at least two points.

Investigation

The Pendulum

Italian mathematician and astronomer Galileo Galilei (1564–1642) made many contributions to our understanding of gravity, the physics of falling objects, and the orbits of the planets. One of his famous experiments involved the periodic motion of a pendulum. In this investigation you will carry out the same experiment and find a function to model the data.

This fresco, painted in 1610, shows Galileo at age 17, contemplating the motion of a swinging lamp in the Cathedral of Pisa. A swinging lamp is an example of a pendulum.

You will need:

- string
- a small weight
- a stopwatch or a watch
- with a second hand

So the horizontal scale factor is 0.28, and the vertical scale factor is −0.42. The negative vertical scale factor also produces a vertical reflection.

Combine these scale factors with the translations to get the final equation

\[
\frac{y - 0.6}{-0.42} = \frac{(x - 0.86)^2}{0.28} \quad \text{or} \quad y = -0.42(x - 0.86)^2 + 0.6
\]

This model, graphed at right, fits the data nicely.

The same procedure works with the other functions you have studied so far. As you continue to add new functions to your mathematical knowledge, you will find that what you have learned about function transformations continues to apply.
ASSESSING PROGRESS
Watch to see how flexible students are at transforming the various functions they are working with. See whether students are able to explain the difference between horizontal and vertical dilations. In a later course, students may see different notation, so urge them to look beyond the symbols and to think about the effect of the dilation on the function as a whole.

DISCUSSING THE INVESTIGATION
To the extent possible, choose students for presenting who obtained different results, especially if they measured in different units. Then have the class look for explanations for the differences.

Step 3  The vertex is at the origin because a pendulum of length 0 cm would have no period.

Step 5  Points farther from the vertex work best. These points represent the longer lengths. They are best for fitting a parabola because they are likely to have less measurement error. A parabola that fits the first few points well would probably be quite far from the points farther from the vertex.

SUPPORT EXAMPLES
1. Describe what happens to the graph of \( y = f(x) \) when it is transformed into \( 3y = f(\frac{x}{2}) \).
   - [vertical dilation (shrink) by a factor of \( \frac{1}{3} \), horizontal dilation (stretch) by a factor of 2]

2. Write an equation for the function that results from translating \( y = |x| \) 3 units left and vertically dilating by a factor of 2. \( y = 2|x + 3| \)

Closing the Lesson

Dilations are nonrigid transformations that expand or shrink graphs horizontally and/or vertically. In an equation, dividing \( x \) by a positive number \( a \) produces an equation of a horizontal dilation by factor \( a \), and dividing \( y \) by positive number \( b \) results in an equation of a vertical dilation by factor \( b \). The dilation is a stretch if the divisor is more than 1; it is a shrink if it’s less than 1. If \( a \) or \( b \) is negative, the graph is reflected across an axis as well as dilated.

[Closing Question] “Why does dividing \( x \) or \( y \) by a number less than 1 result in a shrink?”

[Smaller values of \( x \) or \( y \) will describe the same points on the graph as larger values did in the original equation.]
EXERCISES

Practice Your Skills

1. Each graph is a transformation of the graph of one of the parent functions you’ve studied. Write an equation for each graph.

   ![Graphs of transformations]

   **Exercise 2**
   Describe what happens to the graph of \( y = f(x) \) in these situations.
   
   a. \( x \) is replaced with \( \frac{3}{2} \).
   
   b. \( x \) is replaced with \( -x \).
   
   c. \( x \) is replaced with \( 3x \).
   
   d. \( y \) is replaced with \( \frac{y}{2} \).
   
   e. \( y \) is replaced with \( -y \).
   
   **Exercise 3**
   Solve each equation for \( y \).
   
   a. \( y - 3 = (x - 5)^2 \)
   
   b. \( y + 5 = \left| \frac{x + 3}{3} \right| \)
   
   c. \( y + 7 = \sqrt{\frac{x - 6}{3}} \)

   **Reason and Apply**
   4. For \( b > 0 \), the graphs of \( y = |bx| \) and \( y = |bx| \) are equivalent. For \( b < 0 \), the graph of \( y = |bx| \) is a reflection of \( y = |bx| \) across the \( x \)-axis.

   5. The graph at right shows how to solve the equation \( |x - 4| = 3 \) graphically. The equations \( y = |x - 4| \) and \( y = 3 \) are graphed on the same coordinate axes.
   
   a. What is the \( x \)-coordinate of each point of intersection? What \( x \)-values are solutions of the equation \( |x - 4| = 3 \)?
   
   b. Solve the equation \( |x + 3| = 5 \) algebraically. Verify your solution with a graph.

   **Exercise 4**
   As needed, point out that these are vertical stretches and horizontal shrinks by the same factor (if \( b > 1 \)). *Ask* “Are there other functions for which a vertical stretch by a factor yields the same graph as a horizontal shrink by the same factor?”

   Students can experiment with the parent parabola, the parent square root function, and the parent line. In these three situations, \( \frac{1}{b} \) for \( b > 0 \) is not the same as \( y = \sqrt{bx} \). \( \frac{1}{b} \) is equivalent to \( y = \sqrt{bx} \), but \( \frac{1}{b} = x \) is the same as \( y = bx \).

   **Exercise 5a** *Ask* “Why are there two solutions?” One explanation refers to the graph; another cites the arithmetic; a third gives the two numbers that are 3 units to either side of 4 on the number line.

---

**ASSIGNING EXERCISES**

Suggested Assignments:

<table>
<thead>
<tr>
<th>Type</th>
<th>Assignments</th>
</tr>
</thead>
<tbody>
<tr>
<td>Standard</td>
<td>1, 2, 3, 5, 7, 9, 14</td>
</tr>
<tr>
<td>Enriched</td>
<td>1, 2, 4, 6–11</td>
</tr>
</tbody>
</table>

Types of Exercises:

<table>
<thead>
<tr>
<th>Type</th>
<th>Assignments</th>
</tr>
</thead>
<tbody>
<tr>
<td>Basic</td>
<td>1–3</td>
</tr>
<tr>
<td>Essential</td>
<td>1, 2, 7, 9</td>
</tr>
<tr>
<td>Portfolio</td>
<td>12</td>
</tr>
<tr>
<td>Group</td>
<td>4, 12</td>
</tr>
<tr>
<td>Review</td>
<td>13–14</td>
</tr>
</tbody>
</table>

**EXERCISE NOTES**

If you take two days for this lesson, consider assigning the essential exercises and a review exercise the first day and the other exercises the second day.

**Exercise 1** Students can use their graphing calculators to verify that their equations are correct.

**Exercise 2** Students need not graph these to describe the transformations. *Elaborate* Give students plenty of time to use their new mathematical vocabulary and to discuss their results to this problem.

**Exercise 3** *Extra Support* If students are intimidated by solving equations for \( y \) that have absolute-value and square root symbols, remind them to treat these symbols the same way they treat the parentheses in the quadratic equations. The absolute-value and square root symbols act as grouping symbols in the equations.

**Exercise 3b** Students might state the answer as two separate equations, \( y = \pm \frac{2}{3}x \) and \( y = \pm x - 3 \).

**Exercise 4** As needed, point out that these are vertical stretches and horizontal shrinks by the same factor (if \( b > 1 \)). *Ask* “Are there other functions for which a vertical stretch by a factor yields the same graph as a horizontal shrink by the same factor?”

See page 890 for answers to 1a–p.

226  CHAPTER 4  Functions, Relations, and Transformations
6. **APPLICATION** You can use a single radio receiver to find the distance to a transmitter by measuring the strength of the signal. Suppose these approximate distances are measured with a receiver while you drive along a straight road. Find a model that fits the data. Where do you think the transmitter might be located?

<table>
<thead>
<tr>
<th>Miles traveled</th>
<th>0</th>
<th>4</th>
<th>8</th>
<th>12</th>
<th>16</th>
<th>20</th>
<th>24</th>
<th>28</th>
<th>32</th>
<th>36</th>
</tr>
</thead>
<tbody>
<tr>
<td>Distance from transmitter (miles)</td>
<td>18.4</td>
<td>14.4</td>
<td>10.5</td>
<td>6.6</td>
<td>2.5</td>
<td>1.8</td>
<td>6.0</td>
<td>9.9</td>
<td>13.8</td>
<td>17.6</td>
</tr>
</tbody>
</table>

7. Assume that the parabola \( y = x^2 \) is translated so that its vertex is \((5, -4)\).
   a. If the parabola is dilated vertically by a factor of 2, what are the coordinates of the point on the parabola 1 unit to the right of the vertex? \((6, -2)\)
   b. If the parabola is dilated horizontally instead, by a factor of 3, what are the coordinates of the point on the parabola 1 unit above the vertex? \((2, -3)\) and \((9, -3)\)
   c. If the parabola is dilated vertically by a factor of 2 and horizontally by a factor of 3, name two points on the new parabola that are symmetric with respect to the vertex. \((2, -2)\) and \((8, -2)\)

8. Given the parabola \( y = x^2 \), describe the transformations represented by the function \( y = 4(x - 2)^2 - 5 \). Sketch a graph of the transformed parabola.

9. A curve with parent function \( f(x) = x^2 \) has vertex \((7, 3)\) and passes through the point \((11, 11)\).
   a. What are the values of \( h \) and \( k \) in the equation of the curve? \( h = 7, k = 3 \)
   b. Substitute the values for \( h \) and \( k \) from \( a \) into \( y = k + a \cdot f(x - h) \). Substitute the coordinates of the other point into the equation as values for \( x \) and \( y \).

10. Sketch a graph of each of these equations.
   a. \( \frac{y - 2}{3} = (x - 1)^2 \)
   b. \( \left(\frac{x + 1}{2}\right)^2 = \frac{x - 2}{3} \)
   c. \( \frac{y - 2}{3} = \left|\frac{x + 1}{3}\right| \)

11. Given the graph of \( y = f(x) \), draw graphs of these related functions.

   a. \( y = \frac{1}{2} f(x) \)
   b. \( y = f\left(\frac{x - 3}{2}\right) \)
   c. \( y = \frac{1}{2} f(x + 1) \)

**Exercise 6** [Alert] Students may say that the transmitter is 1.8 mi off the road 20 mi from the starting point. As needed, encourage them to graph the data in order to find the parent function and to write an equation for the transformation.

**Exercise 8** The parabola is dilated vertically by a factor of 3, dilated horizontally by a factor of 4, and translated horizontally \(-7\) units and vertically \(2\) units.

**Exercise 9** Functions like these can also be written so that one of the dilation factors is equal to 1; this makes the equation look less complicated. If \( b = 1 \), then when you solve for \( y \), you find \( y = k + a \cdot f(x - h) \). After finding \( h \) and \( k \), you can find the value of \( a \) by replacing \( x \) and \( y \) with some point from the data and solving for \( a \).

**Exercise 11** [Advanced] If your standards require piecewise functions, you might have students write an equation for \( f(x) \). [assuming the third piece is parabolic, \( y = -2x - 1 \) where \(-3 \leq x \leq -1; y = x + 2 \) where \(-1 \leq x \leq 2; y = 2(x - 3)^2 + 2 \) where \(2 \leq x \leq 3\)]
Exercise 12 [Language] μS is the abbreviation for microsiemens. A siemens is equal to 1 ampere per volt (amp/V). [Context] The conductivity of the solution is directly related to the concentration of ions, independent of their charge. As the acid is added, the concentration of ions decreases as water molecules are formed, until the solution is neutral; it then increases as the solution becomes more acidic. (These data are from Connecting Mathematics with Science: Experiments for Precalculus.)

12. possible equation: \( y = 1050 \cdot (x - 4) + 162 \)

Exercise 13 If students don’t recall how the mean and standard deviation are affected by translations, they might experiment with these data in Fathom. Or use the Science Fair demonstration.

13c. By adding 6 points to each rating, the mean increases by 6, but the standard deviation remains the same.

Exercise 14 Students might do research to compare their predictions for later years with the actual percentages.

14a.

14c. The model predicts 65.1%, so it overestimates by 3.3%.

12. APPLICATION A chemistry class gathered these data on the conductivity of a base solution as acid is added to it. Graph the data and use transformations to find a model to fit the data.

<table>
<thead>
<tr>
<th>Acid volume (mL)</th>
<th>Conductivity (μS/cm²)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>4152.95</td>
</tr>
<tr>
<td>1</td>
<td>3140.97</td>
</tr>
<tr>
<td>2</td>
<td>2100.34</td>
</tr>
<tr>
<td>3</td>
<td>1126.55</td>
</tr>
<tr>
<td>4</td>
<td>162.29</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Acid volume (mL)</th>
<th>Conductivity (μS/cm²)</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>1212.47</td>
</tr>
<tr>
<td>6</td>
<td>2358.11</td>
</tr>
<tr>
<td>7</td>
<td>3417.83</td>
</tr>
<tr>
<td>8</td>
<td>4429.81</td>
</tr>
</tbody>
</table>

Exercise 13 [Language] In 1946, inventors J. Presper Eckert and J. W. Mauchly created the first general-purpose electronic calculator, named ENIAC (Electronic Numerical Integrator and Computer). The calculator filled a large room and required a team of engineers and maintenance technicians to operate it.

2.2 13. A panel of judges rate 20 science fair exhibits as shown. The judges decide that the top rating should be 100, so they add 6 points to each rating.

a. What are the mean and the standard deviation of the ratings before adding 6 points? \( \bar{x} = 83.75, s = 7.45 \)
b. What are the mean and the standard deviation of the ratings after adding 6 points? \( \bar{x} = 89.75, s = 7.45 \)
c. What do you notice about the change in the mean? In the standard deviation?

3.4 14. APPLICATION This table shows the percentage of households with computers in the United States in various years.

<table>
<thead>
<tr>
<th>Year</th>
<th>Households (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1995</td>
<td>31.7</td>
</tr>
<tr>
<td>1996</td>
<td>35.5</td>
</tr>
<tr>
<td>1997</td>
<td>39.2</td>
</tr>
<tr>
<td>1998</td>
<td>42.6</td>
</tr>
<tr>
<td>1999</td>
<td>48.2</td>
</tr>
<tr>
<td>2000</td>
<td>53.0</td>
</tr>
</tbody>
</table>

(www.census.gov)

<table>
<thead>
<tr>
<th>Year</th>
<th>Households (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1995</td>
<td>31.7</td>
</tr>
<tr>
<td>1996</td>
<td>35.5</td>
</tr>
<tr>
<td>1997</td>
<td>39.2</td>
</tr>
<tr>
<td>1998</td>
<td>42.6</td>
</tr>
<tr>
<td>1999</td>
<td>48.2</td>
</tr>
<tr>
<td>2000</td>
<td>53.0</td>
</tr>
</tbody>
</table>

a. Make a scatter plot of these data.
b. Find the median-median line. \( y = 4.23x - 8447.675 \)
c. Compare your model’s prediction for 2003 with the actual census value of 61.8%.
d. Is a linear model for this situation good for long-term predictions? Explain your reasoning.

In 1946, inventors J. Presper Eckert and J. W. Mauchly created the first general-purpose electronic calculator, named ENIAC (Electronic Numerical Integrator and Computer). The calculator filled a large room and required a team of engineers and maintenance technicians to operate it.

14d. Sample answer: A linear model cannot work to predict results for years in the distant future because the percentage cannot increase beyond 100%. There always will be some households without computers, so the long-run percentage will be less than 100%.

EXTENSIONS

A. Use Take Another Look activity 1 or 3 on pages 247 and 248.

B. Have students use a digital camera to take a picture of an object whose shape resembles a parabola or the graph of an absolute-value function. Then have them import it into The Geometer’s Sketchpad, overlay a coordinate grid, and use transformations to plot a function that models the data.

C. Students might collect their own ball-bounce data and repeat Example B.
LESSON 4.7

Transformations and the Circle Family

In this lesson you will investigate transformations of a relation that is not a function. A unit circle is centered at the origin with a radius of 1 unit. Suppose \( P \) is any point on a unit circle with center at the origin. Draw the slope triangle for the radius between the origin and point \( P \).

You can derive the equation of a unit circle from this diagram by using the Pythagorean Theorem. The legs of the right triangle have lengths \( x \) and \( y \) and the length of the hypotenuse is 1 unit, so its equation is \( x^2 + y^2 = 1 \). This is true for all points \( P \) on the unit circle.

What are the domain and the range of this relation? If a value, such as 0.5, is substituted for \( x \), what are the output values of \( y \)? Why is the circle relation not a function?

In order to draw the graph of a circle on your calculator, you need to solve the equation \( x^2 + y^2 = 1 \) for \( y \). When you do this, you get two equations, \( y = \sqrt{1 - x^2} \) and \( y = -\sqrt{1 - x^2} \). Each of these is a function. You have to graph both of them to get the complete circle.

Equation of a Unit Circle
The equation of a unit circle is
\[ x^2 + y^2 = 1 \]
or, solved for \( y \),
\[ y = \pm \sqrt{1 - x^2} \]

You can apply what you have learned about transformations of functions to find the equations of transformations of the unit circle.

EXAMPLE A
Find the equation for each graph.

**DIFFERENTIATING INSTRUCTION**

**ELL**
By this point, all the transformations have been covered. This is a good time to create a graphic organizer, complete with verbal, algebraic, and graphical components of the definition of each transformation.

**Extra Support**
Students may need extra practice solving a standard circle or elliptical equation for \( y \). Students might want to enter their equations into their calculators to verify the location and shape of their graphs.

**Advanced**
Urg students to determine how to transform an equation, whether it is in general form or any other form.

This lesson extends to circles the notions of translation and dilation. If students have not worked with equations of circles before, you may want to spend two days on this lesson. Chapter 8 contains developments in Denmark.

This photo shows circular housing developments in Denmark.
transformations that changed are asked to describe the two formations. Suppose students order for any given set of transformations, then problem-solve if there are discrepancies. When solving for \( y \), they should check both calculator and paper when inputting these equations, to be careful with parentheses.

**LESSON EXAMPLE B**

**[Extra Support]** Students need to be careful with parentheses when inputting these equations, and careful with their algebra when solving for \( y \). They should check both calculator and paper graphs, then problem-solve if there are discrepancies.

Point out that there is a specific order for any given set of transformations. Suppose students are asked to describe the two transformations that changed \( f(x) = x^2 \) into \( g(x) = \left( \frac{x + 2}{3} \right)^2 \).

They should start by looking at how the parent function, \( f(x) \), has been modified. In function form, this equation would look like \( g(x) = f\left( \frac{x + 2}{3} \right) \).

First, \( x \) has been replaced by \( \frac{x}{3} \), representing a horizontal dilation by a factor of 3. Next, \( x \) has been replaced by \( x + 2 \), meaning a horizontal shift 2 units to the left. In this situation, \( f(x) \) was dilated horizontally by a factor of 3 and then shifted horizontally \(-2\) units; each new \( y\)-value is the result of multiplying an \( x\)-value by 3 and then subtracting 2. Note that the order of substitution is not reversible and that this does not follow the logic of the order of operations. If you first replace \( x \) with \( x + 2 \) and then replace \( x \) with \( \frac{x}{3} \), you get \( g(x) = f\left( \frac{x + 2}{3} \right) \). [Big idea] When a graph is transformed, each variable in the graph’s equation is replaced with a variable and a constant representing that aspect of the transformation.

**Guiding the Investigation**

This is a deepening skills investigation.

**Solution**

a. Circle \( a \) is a translation of the unit circle horizontally \(-6\) units and vertically \(2\) units. Replace \( x \) with \( x + 6 \) and \( y \) with \( y - 2 \) to get the equation \((x + 6)^2 + (y - 2)^2 = 1\). To check this result on your calculator, solve for \( y \) and graph:

\[
(y - 2)^2 = 1 - (x + 6)^2 \\
y - 2 = \pm \sqrt{1 - (x + 6)^2} \\
y = 2 \pm \sqrt{1 - (x + 6)^2}
\]

You must enter two functions, \( y = 2 + \sqrt{1 - (x + 6)^2} \) and \( y = 2 - \sqrt{1 - (x + 6)^2} \) into your calculator.

b. Circle \( b \) is a dilation of the unit circle horizontally and vertically by the same scale factor of 3. Replacing \( x \) and \( y \) with \( \frac{x}{3} \) and \( \frac{y}{3} \), you find \( (\frac{x}{3})^2 + (\frac{y}{3})^2 = 1 \). This can also be written as \( \frac{x^2}{9} + \frac{y^2}{9} = 1 \) or \( x^2 + y^2 = 9 \).

You can transform a circle to get an ellipse. An ellipse is a circle where different horizontal and vertical scale factors have been used.

**EXAMPLE B**

What is the equation of this ellipse?

![Ellipse Graph]

**Solution**

The original unit circle has been translated and dilated. The new center is at \((3, 1)\). In a unit circle, every radius measures 1 unit. In this ellipse, a horizontal segment from the center to the ellipse measures 4 units, so the horizontal scale factor is 4. Likewise, a vertical segment from the center to the ellipse measures 3 units, so the vertical scale factor is 3. The equation changes like this:

\[
x^2 + y^2 = 1 \\
\left(\frac{x}{4}\right)^2 + y^2 = 1 \\
\left(\frac{x}{4}\right)^2 + \left(\frac{y}{3}\right)^2 = 1 \\
\frac{(x - 3)^2}{4^2} + \frac{(y - 1)^2}{3^2} = 1
\]

To enter this equation into your calculator to check your answer, you need to solve for \( y \). It takes two equations to graph this on your calculator. By graphing both of these equations, you can draw the complete ellipse and verify your answer.

\[
y = 1 \pm 3\sqrt{1 - \left(\frac{x - 3}{4}\right)^2}
\]
**Investigation**

**When Is a Circle Not a Circle?**

If you look at a circle, like the top rim of a cup, from an angle, you don’t see a circle; you see an ellipse. Choose one of the ellipses from the worksheet. Use your ruler carefully to place axes on the ellipse, and scale your axes in centimeters. Be sure to place the axes so that the longest dimension is parallel to one of the axes. Find the equation to model your ellipse. Graph your equation on your calculator and verify that it creates an ellipse with the same dimensions as on the worksheet.

The tops of these circular oil storage tanks look elliptical when viewed at an angle.

Equations for transformations of relations such as circles and ellipses are sometimes easier to work with in the general form before you solve them for $y$, but you need to solve for $y$ to enter the equations into your calculator. Sometimes, you may need to solve for $y$ to enter the equations into your calculator.

**EXAMPLE C**

If $f(x) = \sqrt{1 - x^2}$, find $g(x) = 2f(3(x - 2)) + 1$. Sketch a graph of this new function.

**Solution**

In $g(x) = 2f(3(x - 2)) + 1$, note that $f(x)$ is the parent function, $x$ has been replaced with $3(x - 2)$, and $f(3(x - 2))$ is then multiplied by 2 and 1 is added.

You can rewrite the function as:

$$g(x) = 2\sqrt{1 - [3(x - 2)]^2} + 1$$

This indicates that the graph of $y = f(x)$, a semicircle, has been dilated vertically by a factor of $\frac{1}{2}$, dilated horizontally by a factor of 2, then translated right 2 units and up 1 unit.

The transformed semicircle is graphed at right. What are the coordinates of the right endpoint of the graph? Describe how the original semicircle’s right endpoint of $(1, 0)$ was mapped to this new location. $(2, 1)$. Multiply the $x$-coordinate by $\frac{1}{2}$ and add 2. Multiply the $y$-coordinate by 2 and add 1.

**MODIFYING THE INVESTIGATION**

**Whole Class** Use the transparency master to draw the ellipse and axes for the class. Challenge students to create the same ellipse on their own calculator, then discuss.

**Shortened** There is no shortened version of this investigation.

**One Step** Hand out the When Is a Circle Not a Circle? worksheet. Ask students to pick one ellipse on the worksheet and find its equation, assuming that the axes are oriented so that the dilation is horizontal or vertical. Ask students to graph the equation on their calculators. As students work, be prepared to refer them back to Lesson 4.6 to see how dilations affect the graphs of the equations of other figures. Remind them as needed how to graph half a unit circle on a calculator.

**FACILITATING STUDENT WORK**

As needed, **[Ask]** “How does the ellipse compare with the circle from which it’s transformed?”

**DISCUSSING THE LESSON**

**LESSON EXAMPLE C**

The coordinates of the right endpoint of the transformed semicircle are now $(2\frac{1}{2}, 1)$. To describe how the original endpoint was mapped to the new location, track the images of $(0, 1)$ and $(1, 0)$ under the various transformations, considering horizontal and vertical dilations.
Lesson Example C (continued)

first. [Alert] Students may think that \( g(x) = 2\sqrt{1 - \left(\frac{x-2}{3}\right)^2} \)
indicates a horizontal shift (to get \( x - 2 \)) before a horizontal
dilation (by factor \( \frac{1}{3} \)). Emphasize
that a transformation is represented by a replacement of variables
in the equation. In this case, \( x \) is first divided by \( \frac{1}{3} \) and then 2
is subtracted.

Start with: \((0, 1)\) \((1, 0)\)
Dilate horizontally
by a factor of \( \frac{1}{3} \): \((0, 1)\) \((\frac{1}{3}, 0)\)
Dilate vertically
by a factor of 2: \((0, 2)\) \((\frac{1}{3}, 0)\)
Translate horizontally
2 units: \((2, 2)\) \((\frac{1}{3}, 0)\)
Translate vertically
1 unit: \((2, 3)\) \((\frac{1}{3}, 1)\)

★ SUPPORT EXAMPLE

Describe the transformation
of the graph of \( y = \sqrt{1 - x^2} \)
needed to produce the graph of
\( y = \sqrt{1 - (2x)^2} + 4 \) [dilated
horizontally by a factor of \( \frac{1}{2} \),
translated vertically 4 units]

Closing the Lesson

Any dilation of a circle is an ellipse (unless you dilate both axes the same amount). To get
an ellipse whose center is not at the origin, you can dilate the unit circle and then translate
the image. The equation of an ellipse is \( \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \). You might
also summarize the relationships between transformed graphs and equations given on this page.

Emphasize that the thinking required to analyze transformations represented by an
equation may be backward from that required to analyze
the order of operations on a variable. [Closing question] “Is a circle an ellipse?” [Yes and no; to get an ellipse, the horizontal and vertical dilation factors must differ. If they don’t differ, it is still a circle.]

You have now learned to translate, reflect, and dilate functions and other relations.
These transformations are the same for all equations.

Transformations of Functions and Other Relations

Translations
The graph of \( y - k = f(x - h) \) translates the graph of \( y = f(x) \)
horizontally \( h \) units and vertically \( k \) units.

or
Replacing \( x \) with \((x - h)\) translates the graph horizontally \( h \) units.
Replacing \( y \) with \((y - k)\) translates the graph vertically \( k \) units.

Reflections
The graph of \( y = f(-x) \) is a reflection of the graph of \( y = f(x) \) across the
\( y \)-axis. The graph of \( -y = f(x) \) is a reflection of the graph of \( y = f(x) \)
across the \( x \)-axis.

or
Replacing \( x \) with \(-x\) reflects the graph across the \( y \)-axis. Replacing \( y \) with \(-y\) reflects the graph across the \( x \)-axis.

Dilations
The graph of \( \frac{y}{a} = f\left(\frac{x}{b}\right) \) is a dilation of the graph of \( y = f(x) \) by a vertical
scale factor of \( a \) and by a horizontal scale factor of \( b \).

or
Replacing \( x \) with \(\frac{x}{b}\) dilates the graph by a horizontal scale factor of \( b \).
Replacing \( y \) with \(\frac{y}{a}\) dilates the graph by a vertical scale factor of \( a \).

Exercises

Practice Your Skills

1. Each equation represents a single transformation. Copy and complete this table.

<table>
<thead>
<tr>
<th>Equation</th>
<th>Transformation (translation, reflection, dilation)</th>
<th>Direction</th>
<th>Amount or scale factor</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y + 3 = x^2 )</td>
<td>Translation Vertical</td>
<td>(-3)</td>
<td></td>
</tr>
<tr>
<td>(-y =</td>
<td>x</td>
<td>)</td>
<td>Reflection Across ( x )-axis</td>
</tr>
<tr>
<td>( y = \sqrt{\frac{x}{4}} )</td>
<td>Dilation Horizontal</td>
<td>4</td>
<td></td>
</tr>
<tr>
<td>( \frac{y}{a} = x^2 )</td>
<td>Dilation Vertical</td>
<td>(0.4)</td>
<td></td>
</tr>
<tr>
<td>( y =</td>
<td>x - 2</td>
<td>)</td>
<td>Translation Horizontal</td>
</tr>
<tr>
<td>( y = \sqrt{\frac{-x}{2}})</td>
<td>Reflection Across ( y )-axis</td>
<td>N/A</td>
<td></td>
</tr>
</tbody>
</table>

Assigning Exercises

Suggested Assignments:
Standard 1, 2, 3, 4a–c, 5, 6, 8, 12
Enriched 4–10, 13, 14

Types of Exercises:
Basic 1–5
Essential 4, 5, 6, 8
Portfolio 9
Group 9
Review 11–16

Exercise Notes

The exercises focus on the semicircle function but also include all the transformations and parent
functions introduced in the chapter. The review exercises are quite lengthy; choose carefully which
ones to assign.

Exercise 1 As needed, encourage students to verify
their ideas by graphing on their calculators.
2. The equation \( y = \sqrt{1 - x^2} \) is the equation of the top half of the unit circle with center \((0, 0)\) shown on the left. What is the equation of the top half of an ellipse shown on the right?

\[
y = 2\sqrt{1 - x^2}
\]

3. Use \( f(x) = \sqrt{1 - x^2} \) to graph each of the transformations below.
   a. \( g(x) = -f(x) \)
   b. \( h(x) = -2f(x) \)
   c. \( j(x) = -3 + 2f(x) \)

4. Each curve is a transformation of the graph of \( y = \sqrt{1 - x^2} \). Write an equation for each curve.
   a. \( \frac{y}{2} = \sqrt{1 - x^2} \) or \( y = 2\sqrt{1 - x^2} \)
   b. \( \frac{x}{5} = \sqrt{1 - x^2} \) or \( y = 0.5\sqrt{1 - x^2} \)
   c. \( \frac{y}{3} = \sqrt{1 - x^2} \) or \( y = 3\sqrt{1 - x^2} \)

5. Write an equation and draw a graph for each transformation of the unit circle. Use the form \( y = \pm\sqrt{1 - x^2} \).
   a. Replace \( y \) with \( y - 2 \).
   b. Replace \( x \) with \( x + 3 \).
   c. Replace \( y \) with \( \frac{x}{2} \)
   d. Replace \( x \) with \( \frac{x}{2} \)

**Reason and Apply**

6. To create the ellipse at right, the \( x \)-coordinate of each point on a unit circle has been multiplied by a factor of 3.
   a. Write the equation of this ellipse. \( \left( \frac{x}{3} \right)^2 + y^2 = 1 \)
   b. What expression did you substitute for \( x \) in the parent equation? \( \frac{x}{3} \)
   c. If \( y = f(x) \) is the function for the top half of a unit circle, then what is the function for the top half of this ellipse, \( y = g(x) \), in terms of \( f(x) \)?

**Exercise 3** If students graph the transformations on paper, suggest that they use two or four squares for each unit.

3a. 
3b. 
3c. 

**Exercise 4** This exercise provides a good way to assess students’ ability to state a transformation algebraically from a given graph. Have students create and proceed through a checklist as they attempt to determine what types of transformations have taken place: Has there been a translation? A dilation? A reflection? Then lead them through making the corresponding algebraic adjustments to the original function.

4c. \( y = \frac{1}{2} \sqrt{1 - x^2} \), or \( y = 2\sqrt{1 - x^2} + 1 \)
4d. \( y = \frac{1}{2} \sqrt{1 - (x - 3)^2} \), or \( y = 2\sqrt{1 - (x - 3)^2} + 1 \)
4e. \( y = \frac{3}{5} \sqrt{1 - \left( \frac{x + 2}{2} \right)^2} \), or \( y = -5\sqrt{1 - \left( \frac{x + 2}{2} \right)^2} + 3 \)
4f. \( y = \frac{2}{4} \sqrt{1 - (x - 3)^2} \), or \( y = 4\sqrt{1 - (x - 3)^2} - 2 \)
5b. \( y = \pm\sqrt{1 - (x + 3)^2} \), or \( (x + 3)^2 + y^2 = 1 \)

5c. \( y = \pm2\sqrt{1 - x^2} \), or \( x^2 + \frac{y^2}{2} = 1 \)
5d. \( y = \pm\sqrt{1 - \left( \frac{x}{2} \right)^2} \), or \( \frac{x^2}{4} + y^2 = 1 \)
Exercise 7 Equivalent forms of the answers include \( x^2 + \left(\frac{y}{3}\right)^2 = 1 \) for 7a, \( \left(\frac{x}{2}\right)^2 + y^2 = 1 \) for 7b, and \( \left(\frac{x}{3}\right)^2 + \left(\frac{y}{1}\right)^2 = 1 \) for 7c.

Exercise 9 This exercise may take a lot of time. Students might benefit from sketching their solutions on graph paper before graphing them on their calculators. The instructions say to imagine drawing a rectangle, but if students want to draw it on their calculators, you can refer them to Calculator Note 41.

9a.

(0, 0) and (1, 1)

9b. The rectangle has width 1 and height 1. The width is the difference in \( x \)-coordinates, and the height is the difference in \( y \)-coordinates.

9c.

(0, 0) and (4, 2)

Exercise 10b–d Students might legitimately think of these functions as either vertical or horizontal dilations of the parent function. The graphs will be the same.

7. Given the unit circle at right, write the equation that generates each transformation. Use the form \( x^2 + y^2 = 1 \).
   a. Each \( y \)-value is half the original \( y \)-value. \( x^2 + \left(\frac{y}{1.5}\right)^2 = 1 \)
   b. Each \( x \)-value is half the original \( x \)-value. \( \left(\frac{x}{4}\right)^2 + y^2 = 1 \)
   c. Each \( y \)-value is half the original \( y \)-value, and each \( x \)-value is twice the original \( x \)-value. \( \left(\frac{x}{2}\right)^2 + (2y)^2 = 1 \)

8. Consider the ellipse at right. 8a. \( y = 3\sqrt{1 - \left(\frac{x}{4}\right)^2} \) and \( y = -3\sqrt{1 - \left(\frac{x}{4}\right)^2} \)
   a. Write two functions that you could use to graph this ellipse.
   b. Use \( \pm \) to write one equation that combines the two equations in 8a.
   c. Write another equation for the ellipse by squaring both sides of the equation in 8b.
   8b. \( y = \pm\sqrt{1 - \left(\frac{x}{4}\right)^2} \) 8c. \( y^2 = 9\left(1 - \left(\frac{x}{4}\right)^2\right) \) or \( \frac{x^2}{16} + \frac{y^2}{9} = 1 \)

9. Mini-investigation Follow these steps to explore a relationship between linear, quadratic, square root, absolute-value, and semicircle functions. Use graphing windows of an appropriate size.
   a. Graph these equations simultaneously on your calculator. The first four functions intersect in the same two points. What are the coordinates of these points?
      \( y = x \quad y = x^2 \quad y = \sqrt{x} \quad y = |x| \quad y = \sqrt{1 - x^2} \)
   b. Imagine using the intersection points that you found in 9a to draw a rectangle that just encloses the quarter-circle that is on the right half of the fifth function. How do the coordinates of the points relate to the dimensions of the rectangle? The rectangle has width 4 and height 2.
   c. Solve these equations for \( y \) and graph them simultaneously on your calculator. Where do the first four functions intersect?
      \( y = \frac{x}{4} \quad y = \left(\frac{x}{2}\right)^2 \quad y = \sqrt{\frac{x}{4}} \quad y = \frac{|x|}{4} \quad y = \sqrt{1 - \left(\frac{x}{4}\right)^2} \)
   d. Imagine using the intersection points that you found in 9c to draw a rectangle that just encloses the right half of the fifth function. How do the coordinates of the points relate to the dimensions of the rectangle?

10. Consider the parent function \( y = \frac{1}{x} \) graphed at right. This function is not defined for \( x = 0 \). When the graph is translated, the center at (0, 0) is translated as well, so you can describe any translation of the figure by describing how the center is transformed.

The parent function passes through the point (1, 1). You can describe any dilations of the function by describing how point (1, 1) is transformed. Use what you have learned about transformations to sketch each graph, then check your work with your graphing calculator.
   a. \( y = \frac{1}{x - 3} \)
   b. \( y + 1 = \frac{1}{x + 4} \)
   c. \( y = \frac{1}{3x} \)
   d. \( \frac{y - 2}{-4} = \frac{1}{x} \)
Exercise 11 Students might use Fathom or a spreadsheet for this exercise.
14a, b.  

Sample answer:  
\[ y = 0.07(x - 3)^2 + 21 \]

Exercise 15 [Alert] Students might not notice that the table skips a couple of intervals in numbers of passengers. [Alert] In 15c, students might add up the right-hand column and divide by something. Or they might add up the means of the intervals in the left-hand column and divide. Help them understand that the number in the right-hand column tells how many airports have a number in the corresponding cell of the left-hand column. The estimate is the sum of the products of the interval means and the number of airports in that interval divided by the total number of airports.

15a, c, d.

(a) Construct a scatter plot of these data.  
(b) Find the equation of a parabola that fits the points and graph it.  
(c) Find the residuals for this equation and the root mean square error.  
(d) Predict the stopping distance for 56.5 mi/h.  
(e) How far off might your prediction in 14d be from the actual stopping distance?

Estimate the mean usage among the 30 airports in 2005.  

a. Display the data in a histogram.  
(b) Estimate the total number of passengers who used the 30 airports. Explain any assumptions you make.  
(c) Estimate the mean usage among the 30 airports in 2005. Mark the mean on your histogram.  
(d) Sketch a box plot above your histogram. Estimate the five-number summary values. Explain any assumptions you make. 

Five-number summary: 32.5, 32.5, 42.5, 52.5, 87.5; assume that all data occur at midpoints of bins.

2.3 16. Consider the linear function \( y = 3x + 1 \).

(a) Write an equation for the image of the graph of \( y = 3x + 1 \) after a reflection across the \( x \)-axis. Graph both lines on the same axes.  
(b) Write an equation for the image of the graph of \( y = 3x + 1 \) after a reflection across the \( y \)-axis. Graph both lines on the same axes.  
(c) Write an equation for the image of the graph of \( y = 3x + 1 \) after a reflection across the \( x \)-axis and then across the \( y \)-axis. Graph both lines on the same axes.  
(d) How does the image in 16c compare to the original line? The two lines are parallel.

IMPROVING YOUR VISUAL THINKING SKILLS

4-in-1  

Copy this trapezoid. Divide it into four congruent polygons.
Compositions of Functions

Sometimes you’ll need two or more functions in order to answer a question or analyze a problem. Suppose an offshore oil well is leaking. Graph A shows the radius, \( r \), of the spreading oil slick, growing as a function of time, \( t \), so \( r = f(t) \). Graph B shows the area, \( a \), of the circular oil slick as a function of its radius, \( r \), so \( a = g(r) \). Time is measured in hours, the radius is measured in kilometers, and the area is measured in square kilometers.

Suppose you want to find the area of the oil slick after 4 hours. You can use function \( f \) on Graph A to find that when \( t \) equals 4, \( r \) equals 1.5. Next, using function \( g \) on Graph B, you find that when \( r \) equals 1.5, \( a \) is approximately 7. So, after 4 h, the radius of the oil slick is 1.5 km and its area is 7 km\(^2\).

You used the graphs of two different functions, \( f \) and \( g \), to find that after 4 h, the oil slick has area 7 km\(^2\). You actually used the output from one function, \( f \), as the input in the other function, \( g \). This is an example of a composition of functions to form a new functional relationship between area and time, that is, \( a = g(f(t)) \).

The symbol \( g(f(t)) \), read “\( g \) of \( f \) of \( t \)” is a composition of the two functions \( f \) and \( g \). The composition \( g(f(t)) \) gives the final outcome when an \( x \)-value is substituted into the “inner” function, \( f \), and its output value, \( f(t) \), is then substituted as the input into the “outer” function, \( g \).

**EXAMPLE A**

Consider these functions:

\[
\begin{align*}
    f(x) &= \frac{3}{4}x - 3 \\
    g(x) &= |x|
\end{align*}
\]

What will the graph of \( y = g(f(x)) \) look like?

**Solution**

Function \( f \) is the inner function, and function \( g \) is the outer function. Use equations and tables to identify the output of \( f \) and use it as the input of \( g \).
Find several \( f(x) \) output values. Use the \( f(x) \) output values as the input of \( g(x) \). Match the input of the inner function, \( f \), with the output of the outer function, \( g \), and plot the graph.

\[
\begin{array}{c|c|c|c|c|c}
 x & f(x) & f(x) & g(f(x)) & f(x) & g(f(x)) \\
-2 & -4.5 & -4.5 & 4.5 & -2 & 4.5 \\
0 & -3 & -3 & 3 & 0 & 3 \\
2 & -1.5 & -1.5 & 1.5 & 2 & 1.5 \\
4 & 0 & 0 & 0 & 4 & 0 \\
6 & 1.5 & 1.5 & 1.5 & 6 & 1.5 \\
8 & 3 & 3 & 3 & 8 & 3 \\
\end{array}
\]

The solution is the composition graph at right. All the function values of \( f \), whether positive or negative, give positive output values under the rule of \( g \), the absolute-value function. So, the part of the graph of function \( f \) showing negative output values is reflected across the \( x \)-axis in this composition.

You can use what you know about transformations to get the specific equation for \( y = g(f(x)) \) in Example A. Use the parent function \( y = |x| \), translate the vertex right 4 units, and then dilate horizontally by a factor of 4 and vertically by a factor of 3. This gives the equation \( y = \frac{1}{3}(x-4) \). You can algebraically manipulate this equation to get the equivalent equation \( y = \frac{1}{2}|x-3| \), which is the equation of \( f \) substituted for the input of \( g \). You can always create equations of composed functions by substituting one equation into another.

**You will need**
- a small mirror
- one or more tape measures or metersticks

**Procedure Note**
1. Place the mirror flat on the floor at least 0.5 m from a wall.
2. Use tape to attach tape measures or metersticks up the wall to a height of 1.5 to 2 m.

**Step 1**
Set up the experiment as in the Procedure Note. Stand a short distance from the mirror, and look down into it. Move slightly left or right until you can see the tape measure on the wall reflected in the mirror.

**Step 2**
Have a group member slide his or her finger up the wall to help locate the highest height mark that is reflected in the mirror. Record the height in centimeters, \( h \), and the distance from your toe to the center of the mirror in centimeters, \( d \).

You need additional equipment to complete this activity. Four students collect data for Students A through D.

**Guiding the Investigation**
This is a deepening skills investigation. It is also an activity investigation. You can use the sample data if you do not wish to conduct the investigation as an activity.

**MODIFYING THE INVESTIGATION**

**Whole Class** Have four students collect data for the class, or demonstrate data collection and use sample data. Complete Steps 4 through 8 with student input.

**Shortened** Use the sample data. Discuss Steps 4 through 8 as a class.

**One Step** Hand out materials and pose this problem: “Attach the tape measure (or metersticks) on the wall from the floor up to a height of from 1.5 to
### LESSON 4.8 Compositions of Functions

#### Step 3
**Step 3 sample data:** (50, 148), (70, 106), (100, 73.5), (130, 57), (160, 45)

Change your distance from the mirror and repeat Step 2. Make sure you keep your head in the same position. Collect several pairs of data in the form \((d, h)\). Include some distances from the mirror that are small and some that are large.

#### Step 4
Find a function that fits your data by transforming the parent function \(h = \frac{1}{2}t^2\). Call this function \(f\). Possible answer: \(h = f(t) = \frac{7400}{d}\)

Now you’ll combine your work from Steps 1–4 with the scenario of a timed walk toward and away from the mirror.

#### Step 5
**Step 5 possible answer:**
\[ d = g(t) = 82\left(\frac{2x-4.7}{3.5}\right)^2 + 30 \]

Suppose this table gives your position at 1-second intervals:

<table>
<thead>
<tr>
<th>Time (s) (t)</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>Distance to mirror (cm) (d)</td>
<td>163</td>
<td>112</td>
<td>74</td>
<td>47</td>
<td>33</td>
<td>31</td>
<td>40</td>
<td>62</td>
</tr>
</tbody>
</table>

Use one of the families of functions from this chapter to fit these data. Call this function \(g\). It should give the distance from the mirror for seconds 0 to 7.

#### Step 6
**Step 6 Answers will vary.** These answers are based on the sample data from Step 3.

a. Where are you at 1.3 seconds? 105 cm from mirror
b. How high up the wall you can see when you are 60 cm from the mirror; 123 cm

c. How high up the wall can you see at 3.4 seconds? 178 cm

#### Step 7
Change each expression into words relating to the context of this investigation and find an answer. Show the steps you needed to evaluate each expression.

- a. \(f(60)\) how high up the wall you can see when you are 60 cm from the mirror; 123 cm
- b. \(g(5.1)\) your distance from the mirror at 5.1 s; 31 cm
- c. \(f(g(2.8))\) how high you can see up the wall at 2.8 s; 137 cm

#### Step 8
Find a single function, \(H(t)\), that does the work of \(f(g(t))\). Show that \(H(2.8)\) gives the same answer as Step 7c above. For sample data: \(H(t) = \frac{7400}{82(\frac{2x-4.7}{3.5})^2 + 30}\)

Don’t confuse a composition of functions with the product of functions. In Example A, you saw that the composition of functions \(f(x) = \frac{1}{2}x - 3\) and \(g(x) = |x|\) is \(g(f(x)) = |\frac{1}{2}x - 3|\). However, the product of the functions is \(f(x) \cdot g(x) = \left(\frac{1}{2}x - 3\right) \cdot |x|\), or \(|x|/2x - 3|x|\). Multiplication of functions is commutative, so \(f(x) \cdot g(x) = g(x) \cdot f(x)\).

### FACILITATING STUDENT WORK

**Steps 1–3** Make sure the student who is looking into the mirror maintains the same head height and position throughout the data collection.

**Steps 4** Student equations may vary. Ask them to check their equation against all their data before they continue to Step 5.

**Steps 5** You might suggest that students first graph the data and then use transformations to find the equation. Their equations of the parabola may vary from group to group. Using the calculator techniques discussed in Lesson 4.6 and Calculator Note 4G, students can translate the parent graph \(y = x^2\) so it coincides with the graph. The vertex is not located at a data point, so students will need to estimate its coordinates. The answers have been calculated based on an estimated vertex at \((4.7, 30)\), and using the point \((1, 112)\) to determine the \(a\)-value.
DISCOVERING ADVANCED ALGEBRA COURSE SAMPLER

ASSESSING PROGRESS
Check that students are not turning $f(g(x))$ into a multiplication problem. These exercises are otherwise very straightforward. If students are able to find a solution, they have a baseline understanding. Students who show a more advanced understanding can describe the behavior of each function in the composition.

DISCUSSING THE INVESTIGATION
To the extent possible, have students present a variety of equations from Step 8 of the investigation. Keep asking why the results differ. Data-collection procedures and differences in assumptions about translations and dilations will account for some differences. Errors in calculation and in students’ understanding of composition will account for others.

[Critical Question] “How do you know which function is substituted into which when you’re finding the composition?” [Big Idea] You use the meaning of function notation. If you are trying to find $f(g(x))$, you substitute the expression of function $g$ for the variable in the expression of function $f$.

LESSON EXAMPLE B
[Alert] Students may find it confusing to consider the domain of the inner function twice. A non-symbolic example might help. For example, suppose you’re at a party at which everyone takes off his or her shoes. There’s a function from the people to their shoes and another function from the shoes to their numerical sizes. But not all shoes have numerical sizes, so the domain of the second function is limited. To find the domain of the composition, you need to go back to the people and determine who was wearing shoes with numerical sizes.

EXERCISES

Practice Your Skills

1. Given the functions $f(x) = 3 + \sqrt{x + 5}$ and $g(x) = 2 + (x - 1)^2$, find these values.
   a. $f(4)$  
   b. $f(g(4))$  
   c. $g(-1)$  
   d. $g(f(-1))$  

SUPPORT EXAMPLE
If $f(x) = \frac{1}{x+1}$ and $g(x) = x^2 + 5$, find $f(g(3))$ and $g(f(-1))$. $[\frac{1}{13}, 46, 9]$  

Closing the Lesson

The main point of this lesson is that the composition of two functions $f(x)$ and $g(x)$ is a new function $g(f(x))$ that takes the output of $f(x)$ as input to $g(x)$.

Composing functions requires you to replace the independent variable in one function with the output value of the other function. This means that it is generally not commutative. That is, $f(g(x)) \neq g(f(x))$, except for certain functions.

To find the domain and range of a composite function, you must look closely at the domain and range of the original functions.

EXAMPLE B
Let $f(x)$ and $g(x)$ be the functions graphed below. What is the domain of $f(g(x))$?

Solution
Start by identifying the domain of the inner function, $g(x)$. This domain, as seen on the graph, is $1 \leq x \leq 5$. These values produce a range of $1 \leq g(x) \leq 3$. This is the input for the outer function, $f(x)$. However, notice that not all of these output values lie in the domain of $f(x)$. For example, there is no value for $f(2.5)$. Only the values $1 \leq g(x) \leq 2$ are in the domain of $f$. Now identify the $x$-values that produced this part of the range of $g(x)$. This is the domain of the composite function. The domain is $1 \leq x \leq 3$.  

To find the domain of a composite function, first use the domain of the inner function to find its range. Then find the subset of the range that is within the domain of the outer function. The $x$-values that produce that subset of values are the domain of the composite function.

[Closing Question] “If $f(x) = -4x + 1$ and $g(x) = x^2$, what’s the difference between $f(g(x))$ and $g(f(x))$?” Answers may vary; students may simply say that $f(g(x)) = -4x^2 + 1$ and $g(f(x)) = (-4x + 1)^2$. Or they may try a more complex comparison, such as mentioning a binomial and a trinomial.

240  CHAPTER 4 Functions, Relations, and Transformations

DISCOVERING ADVANCED ALGEBRA COURSE SAMPLER

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2. The functions \( f \) and \( g \) are defined by these sets of input and output values.

\[
g = \{(1, 2), (-2, 4), (5, 5), (6, -2)\}
\]
\[
f = \{(0, -2), (4, 1), (3, 5), (5, 0)\}
\]

a. Find \( g(f(4)) \).  

b. Find \( f(g(-2)) \).  

c. Find \( f(g(f(3))) \).

3. APPLICATION Graph A shows a swimmer’s speed as a function of time. Graph B shows the swimmer’s oxygen consumption as a function of her speed. Time is measured in seconds, speed in meters per second, and oxygen consumption in liters per minute. Use the graphs to estimate the values.

![Graph A and Graph B](image)

a. the swimmer’s speed after 20 s of swimming approximately 1.5 m/s  
b. the swimmer’s oxygen consumption at a swimming speed of 1.5 m/s approximately 12 L/min  
c. the swimmer’s oxygen consumption after 40 s of swimming approximately 15 L/min

4. Identify each equation as a composition of functions, a product of functions, or neither. If it is a composition or a product, then identify the two functions that combine to create the equation.

\[
a. y = 5\sqrt{x} + 2x
\]
\[
b. y = 3 + \frac{1}{2} \cdot (x + 5) - 3^2
\]
\[
c. y = (x - 5)^2 \cdot 2 - \sqrt{x}
\]

5. Consider the graph at right.

a. Write an equation for this graph: \( y = (x - 3)^3 - 1 \)

b. Write two functions, \( f \) and \( g \), such that the figure is the graph of \( y = f(g(x)) \). \( f(x) = x \) and \( g(x) = (x - 3)^3 - 1 \)

6. The functions \( f \) and \( g \) are defined by these sets of input and output values:

\[
g = \{(1, 2), (-2, 4), (5, 5), (6, -2)\}
\]
\[
f = \{(2, 1), (4, -2), (5, 5), (-2, 6)\}
\]

a. Find \( g(f(2)) \).  

b. Find \( f(g(6)) \).

6c. Select any number from the domain of either \( g \) or \( f \), and find \( f(g(x)) \) or \( g(f(x)) \), respectively. Describe what is happening. The composition of \( f \) and \( g \) will always give back the original number because \( f \) and \( g \) “undo” the effects of each other.

Exercise 2 As needed, remind students that relations, including functions, can be thought of as sets of ordered pairs. In this case, \( g(1) = 2 \) and \( f(4) = 1 \). You may need to help students realize that they should evaluate the inner function first and work out from there.

Exercise 4 Different answers are possible. For example, the function in 4b can also be considered as \( g(f(x)) \) for \( g(x) = 3 + x \) and \( f(x) = (x + 5) - 3^2 \). [Extra Support] This exercise provides students with the opportunity to determine the difference between the product and the composition of two functions. After students find the two functions used to create the composition, suggest that they perform the composition to verify their results.

Exercise 5 If students are having difficulty, you might suggest that they consider what would happen if instead of changing direction suddenly, the graph continued along a vertical reflection of the middle section. [Advanced] This exercise provides students with the challenge of finding a composition function that would create the given graph.

Exercise 6c If students do not include a reason in their answer, [Ask] “Why?” [The pairs in function \( g \) are the reverse of the pairs in function \( f \).]
7. A, B, and C are gauges with different linear measurement scales. When A measures 12, B measures 13, and when A measures 36, B measures 29. When B measures 20, C measures 57, and when B measures 32, C measures 84.
   a. Sketch separate graphs for readings of B as a function of A and readings of C as a function of B. Label the axes.
   b. If A reads 12, what does C read? 
   c. Write a function with the reading of the A as the dependent variable and the reading of the B as the independent variable.
   d. Write a function with the reading of B as the dependent variable and the reading of C as the independent variable.
   e. Write a function with the reading of C as the dependent variable and the reading of A as the independent variable.

8. The graph of the function $y = g(x)$ is shown at right. Draw a graph of each of these related functions.
   a. $y = \sqrt{g(x)}$
   b. $y = -g(x)$
   c. $y = (g(x))^2$
   d. What is the domain of each function in 8a–c?
      a. $x \geq -2$; b. all real numbers; c. all real numbers

9. The two lines pictured at right are $f(x) = 2x - 1$ and $g(x) = \frac{3}{2}x + \frac{1}{2}$. Solve each problem both graphically and numerically.
   a. Find $g(f(2))$. 
   b. Find $f(g(1))$. 
   c. Pick your own $x$-value in the domain of $f$, and find $g(f(x))$.
   d. Pick your own $x$-value in the domain of $g$, and find $f(g(x))$.
   e. Carefully describe what is happening in these compositions.
   The two functions “undo” the effects of one another and thus give back the original value.

10. Given the functions $f(x) = -x^2 + 2x + 3$ and $g(x) = (x - 2)^2$, find these values.
    a. $f(g(3))$ 
    b. $f(g(2))$ 
    c. $g(f(0.5))$ 
    d. $g(f(1))$ 
    e. $f(g(x))$. Simplify to remove all parentheses.
    f. $g(f(x))$. Simplify to remove all parentheses.
    Include a See Calculator Note 4H to learn how to use your calculator to check the answers to 10e and 10f.

11. Aaron and Davis need to write the equation that will produce the graph at right.
    Aaron: “This is impossible! How are we supposed to know if the parent function is a parabola or a semicircle? If we don’t know the parent function, there is no way to write the equation.”
    Davis: “Don’t panic yet. I am sure we can determine its parent function if we study the graph carefully.”
    Do you agree with Davis? Explain completely and, if possible, write the equation of the graph.

9c. $g(f(x)) = g(2x - 1) = \frac{3}{2}(2x - 1) + \frac{1}{2} = x$ for all $x$
9d. $f(g(x)) = f(\frac{3}{2}x + \frac{1}{2}) = \frac{3}{2}(\frac{3}{2}x + \frac{1}{2}) - 1 = x$ for all $x$

Exercise 9e If students do not include a reason in their answer, [Ask] “Why?”

11. If the parent function is $y = x^2$, then the equation is $y = -3x^2 + 3$. If the parent function is $y = \sqrt{1 - x^2}$, then the equation is $y = 3\sqrt{1 - x^2}$. It appears that when $x = 0.5$, $y \approx 2.6$.
    Substituting 0.5 for $x$ in each equation gives the following results:
    $-3(0.5)^2 + 3 = 2.25$
    $3\sqrt{1 - 0.5^2} \approx 2.598$
    Thus, the stretched semicircle is the better fit.
12. **APPLICATION** Jen and Priya decide to go out to the Hamburger Shack for lunch. They each have a $0.50 cent coupon from the Sunday newspaper for the Super-Duper-Deluxe $5.49 Value Meal. In addition, if they show their I.D. cards, they’ll also get a 10% discount. Jen’s server rang up the order as Value Meal, coupon, and then I.D. discount. Priya’s server rang it up as Value Meal, I.D. discount, and then coupon.

a. How much did each girl pay? Jen: $4.49; Priya: $4.44
b. Write a function, \( C(x) \), that will deduct 50 cents from a price, \( x \).\[ C(x) = x - 0.50 \]
c. Write a function, \( D(x) \), that will take 10% off a price, \( x \).\[ D(x) = 0.90x \]
d. Find \( C(D(x)) \).\[ C(D(x)) = 0.90x - 0.50 \]
e. Which server used \( C(D(x)) \) to calculate the price of the meal? Priya’s server
f. Is there a price for the Value Meal that would result in both girls paying the same price? If so, what is it? There is no price because \( 0.90x - 0.50 = 0.90(x - 0.50) \) has no solution.

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### Review

13. Solve.\( \sqrt{x - 4} = 3 \)
   a. \( x = -5 \) or \( x = 13 \)
   b. \( (3 - \sqrt{x + 2})^2 = 4 \)
   c. \( |3 - \sqrt{x}| = 5 \)
   d. \( 3 + 5\sqrt{1 + 2x^2} = 13 \)

14. **APPLICATION** Bonnie and Mike are working on a physics project. They need to determine the ohm rating of a resistor. The ohm rating is found by measuring the potential difference in volts and dividing it by the electric current, measured in amperes (amps). In their project they set up the circuit at right. They vary the voltage and observe the corresponding readings of electrical current measured on the ammeter.

<table>
<thead>
<tr>
<th>Potential difference (volts)</th>
<th>12</th>
<th>10</th>
<th>6</th>
<th>4</th>
<th>3</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>Current (amps)</td>
<td>2.8</td>
<td>2.1</td>
<td>1.4</td>
<td>1.0</td>
<td>0.6</td>
<td>0.2</td>
</tr>
</tbody>
</table>

a. Identify the independent and dependent variables.
b. Display the data on a graph.
c. Find the median-median line. \( y = 0.2278x + 0.0167 \)
d. Bonnie and Mike reason that because 0 volts obviously yields 0 amps, the line they really want is the median-median line translated to go through \((0, 0)\). What is the equation of the line through the origin that is parallel to the median-median line? \( y = 0.2278x \)
e. How is the ohm rating Bonnie and Mike are trying to determine related to the line in 14d? \( y \) is the reciprocal of the slope of this line.
f. What is their best guess of the ohm rating to the nearest tenth of an ohm? \( 4.4 \) ohms

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**Exercise 12** As an extension, ask “What would happen if the Sunday coupon were for a 5% discount instead of 50 cents off?” Another extension might involve a store that has items on 50% off clearance, followed by an additional 20% discount for a holiday sale and an additional 15% off if you use the store’s credit card. Ask “Is the total discount 85%?” [No; the cost is \( 0.85(0.8(0.5p)) = 0.34p \), or a discount of 66%.

**Exercise 13** Although students may previously have solved equations like these with graphing, encourage them to solve the equations symbolically here. Ask “How do you deal with the two values located within the absolute-value symbol?” [Write two equations.]

**Exercise 14** Students who don’t understand electricity very well may be intimidated by this problem. Assure them that they can solve it if they understand the mathematics. Praise success at overcoming the psychological barrier.

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LESSON 4.8 Compositions of Functions
EXTENSION

Use Take Another Look activities 4 and 5 on page 248.

15b.

<table>
<thead>
<tr>
<th>x</th>
<th>y</th>
</tr>
</thead>
<tbody>
<tr>
<td>(-3, 0)</td>
<td>(0, 3)</td>
</tr>
<tr>
<td>(0, -3)</td>
<td>(3, 0)</td>
</tr>
</tbody>
</table>

Encourage students to experiment with the graph until they understand how each function contributes to the picture.

4.7 15. Begin with the equation of the unit circle, \(x^2 + y^2 = 1\).
   a. Apply a horizontal dilation by a factor of 3 and a vertical dilation by a factor of 3, and write the equation that results. \(\left(\frac{x}{3}\right)^2 + \left(\frac{y}{3}\right)^2 = 1\), or \(x^2 + y^2 = 9\)
   b. Sketch the graph. Label the intercepts.

4.4 16. Imagine translating the graph of \(f(x) = x^3\) left 3 units and up 5 units, and call the image \(g(x)\).
   a. Give the equation for \(g(x)\). \(g(x) = (x + 3)^3 + 5\)
   b. What is the vertex of the graph of \(y = g(x)\)? \((-3, 5)\)
   c. Give the coordinates of the image point on the parabola that is 2 units to the right of the vertex. \((-1, 9)\)

### Project

**件wise Pictures**

You can use piecewise functions to create designs and pictures. If you use several different functions together, you can create a picture that does not represent a function. See Calculator Note 4E to learn more about graphing piecewise functions.

You can use your calculator to draw this car by entering these functions:

- \(c(x) = \begin{cases} 1 + 1.2\sqrt{x - 1}, & x \leq 3.5 \\ 4 - 0.5(x - 5)^2, & 3.5 \leq x \leq 6.5 \\ 1 + 1.2\sqrt{(x - 9)}, & 6.5 \leq x \end{cases}\)
- \(d(x) = \begin{cases} 1, & 1 \leq x \leq 9 \\ 0, & 9 < x \end{cases}\)
- \(e(x) = 1 - \sqrt{1 - (x - 2.5)^2}\)
- \(g(x) = 1 - \sqrt{1 - (x - 7.5)^2}\)
- \(h(x) = \begin{cases} 2 + |x - 5.5|, & 5.2 \leq x \leq 5.8 \\ 0, & \text{otherwise} \end{cases}\)

Which function represents which part of the car? Explain why some of the functions do not have restricted domains.

Experiment with the given piecewise functions to see if you can modify the shape of the car or increase its size. Then write your own set of functions to draw a picture. Your project should include:
- A screen capture or accurate graph of your drawing.
- The functions you used to create your drawing, including any restrictions on the domain.
- At least one piecewise function.

### Supporting the Project

Encourage students to experiment with the graph until they understand how each function contributes to the picture.

### OUTCOMES

- The screen capture shows a drawing that uses several transformations of parent functions.
- At least one function used is piecewise.
- Optional: The student produces a complex drawing that uses at least one of each function appearing in this chapter.
This chapter introduced the concept of a function and reviewed function notation. You saw real-world situations represented by rules, sets, functions, graphs, and most importantly, equations. You learned to distinguish between functions and other relations by using either the definition of a function—at most one y-value per x-value—or the vertical line test.

This chapter also introduced several transformations, including translations, reflections, and vertical and horizontal dilations. You learned how to transform the graphs of parent functions to investigate several families of functions—linear, quadratic, square root, absolute value, and semicircle. For example, if you dilate the graph of the parent function \( y = x^2 \) vertically by a factor of 3 and horizontally by a factor of 2, and translate it right 1 unit and up 4 units, then you get the graph of the function \( y = 3(\frac{x}{2})^2 + 4 \).

Finally, you looked at the composition of functions. Many times, solving a problem involves two or more related functions. You can find the value of a composition of functions by using algebraic or numeric methods or by graphing.

**EXERCISES**

Answers are provided for all exercises in this set.

1. Sketch a graph that shows the relationship between the time in seconds after you start microwaving a bag of popcorn and the number of pops per second. Describe in words what your graph shows.

2. Use these three functions to find each value:
   - \( f(x) = -2x + 7 \)
   - \( g(x) = x^2 - 2 \)
   - \( h(x) = (x + 1)^2 \)
   a. \( f(4) = ? \)
   b. \( g(-3) = ? \)
   c. \( h(x + 2) = ? \) \( (x + 3)^2 - 3 \)
   d. \( f(g(x)) = ? \)
   e. \( g(h(-2)) = ? \)
   f. \( h(f(a)) = ? \)
   g. \( f(g(a)) = ? \)
   h. \( g(f(a)) = ? \)
   i. \( f(g(a)) = ? \)
   j. \( g(f(a)) = ? \)
   k. \( f(g(a)) = ? \)
   l. \( g(f(a)) = ? \)

3. The graph of \( y = f(x) \) is shown at right. Sketch the graph of each of these functions:
   a. \( y = f(x - 3) \)
   b. \( y = f(x - 3) \)
   c. \( y = f(x) \)
   d. \( y = f(-x) \)

**ASSIGNING EXERCISES**

If you are using one day to review this chapter, limit the number of exercises you assign. Several of the exercises have many parts.

**EXERCISE NOTES**

Exercise 2 As needed, remind students that \( f(g(a)) \) is not necessarily the same as \( g(f(a)) \) and that the result of evaluating these functions at \( a \) will not be a number.

Exercise 3 You might pass out copies of the Exercises 3 and 5 worksheet and let students graph and label the answers on the worksheet.
4. Assume you know the graph of \( y = f(x) \). Describe the transformations, in order, that would give you the graph of these functions:
   a. \( y = f(x + 2) - 3 \)
   b. \( y = \frac{1}{2} f(x) + 1 \)
   c. \( y = 2 f\left(\frac{x - 1}{0.5}\right) + 3 \)

4a. Translate horizontally \(-2\) units and vertically \(-3\) units.

5. The graph of \( y = f(x) \) is shown at right. Use what you know about transformations to sketch these related functions:
   a. \( y - 1 = f(x - 2) \)
   b. \( y + \frac{3}{2} = f(x + 1) \)
   c. \( y = f(-x) + 1 \)
   d. \( y + 2 = f\left(\frac{x}{2}\right) \)
   e. \( y = -f(x - 3) + 1 \)
   f. \( y + \frac{2}{-2} = f\left(\frac{x - 1}{1.5}\right) \)

6. For each graph, name the parent function and write an equation of the graph.
   a. \( y = \sqrt{1 - x^2}; y = 2\sqrt{1 - \left(\frac{x}{2}\right)^2} + 3 \)
   b. \( y = \sqrt{1 - x^2}; y = 4\sqrt{1 - \left(\frac{x - 3}{2}\right)^2} - 1 \)
   c. \( y = \sqrt{1 - x^2}; y = \sqrt{x}; y = -\sqrt{-(x - 2) - 3} \)
   d. \( y = x^2; y = (x - 2)^2 - 4 \)
   e. \( y = x^2; y = x - 2(x + 1)^2 \)
   f. \( y = x^2; y = x^2 - 2(x + 1)^2 \)

7. Solve for \( y \).
   a. \( 2x - 3y = 6 \) \( y = \frac{2x}{3} - 2 \)
   b. \( (y + 1)^2 - 3 = x \) \( y = \pm \sqrt{x + 3} - 1 \)
   c. \( \sqrt{1 - y^2} + 2 = x \) \( y = \pm \sqrt{(x - 2)^2} + 1 \)

8. Solve for \( x \).
   a. \( 4\sqrt{x} - x = 10 \) \( x = 8.25 \)
   b. \( \left(\frac{x}{2}\right)^2 = 5 \) \( x = \pm 2\sqrt{5} \approx 6.7 \)
   c. \( \frac{x - 2}{2} - 4 \) \( x = 11 \) or \( x = -5 \)
   d. \( \frac{y}{4} + \frac{3}{5} = 2 \) no solution.
9. The Acme Bus Company has a daily ridership of 18,000 passengers and charges $1.00 per ride. The company wants to raise the fare yet keep its revenue as large as possible. (The revenue is found by multiplying the number of passengers by the fare charged.) From previous fare increases, the company estimates that for each increase of $0.10 it will lose 1,000 riders.

a. Complete this table.

<table>
<thead>
<tr>
<th>Fare ($)</th>
<th>1.00</th>
<th>1.10</th>
<th>1.20</th>
<th>1.30</th>
<th>1.40</th>
<th>1.50</th>
<th>1.60</th>
<th>1.70</th>
<th>1.80</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of passengers</td>
<td>18,000</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Revenue ($)</td>
<td>18,000</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

b. Make a graph of the revenue versus fare charged. You should recognize the graph as a parabola.

c. What are the coordinates of the vertex of the parabola? Explain the meaning of each coordinate of the vertex.

d. Find a quadratic function that models these data. Use your model to find

i. the revenue if the fare is $2.00.

$16,000

ii. the fare(s) that make no revenue ($0).

$0 or $2.80

iii. the fare(s) that make maximum revenue, $19,600.

$1.40

TAKE ANOTHER LOOK

1. Some functions can be described as even or odd. An even function has the y-axis as a line of symmetry. If the function f is an even function, then f(x) = f(x) for all values of x in the domain. Which parent functions that you’ve seen are even functions? Now graph y = x^2, y = \frac{1}{x}, and y = \sqrt{x}, all of which are odd functions. Describe the symmetry displayed by these odd functions. How would you define an odd function in terms of f(x)? If possible, give an example of a function that is neither even nor odd.

2. A line of reflection does not have to be the x- or y-axis. Draw the graph of a function and then draw its image when reflected across different horizontal or vertical lines. Write the equation of each image. Try this with several different functions. In general, if the graph of y = f(x) is reflected across the vertical line x = a, what is the equation of the image? If the graph of y = f(x) is reflected across the horizontal line y = b, what is the equation of the image?

Exercise 9d [Ask] “Why, in real life, would neither of these fares result in revenue?” [For $0, you would be charging no fare, so you would take in no revenue. For $2.80, the fare is so expensive that no passengers would take the bus.]

TAKE ANOTHER LOOK

1. The parent functions y = x^2 and y = |x| are even functions. An odd function is said to have symmetry with respect to the origin. Students might also describe it as twofold rotational symmetry (through 180°). If the function f is an odd function, then f(x) = -f(x) for all values of x in the domain. The linear function y = a + bx is an example of a function that is neither even nor odd when a ≠ 0 and b ≠ 0.

2. Reflecting the graph across the vertical line x = a is equivalent to translating the graph horizontally by the amount a (to move the line x = a to the y-axis), reflecting it across the y-axis, and then translating it back. This composition of transformations yields the equation y = f(-x - a) + a = f(-x + 2a). By a similar composition, a reflection across the horizontal line y = b is given by the equation y = -(f(x) - b) + b = -f(x) + 2b.
3. The semicircle function, \( y = \sqrt{1 - x^2} \), and the circle relation, \( x^2 + y^2 = 1 \), are two examples for which a vertical dilation is not equivalent to any horizontal dilation.

4. The graphs of the compositions of any two linear equations will be parallel. The linear equations resulting from the compositions will have the same slope, or \( x \)-coefficient.

Algebraic proof:
Let \( f(x) = ax + b \) and \( g(x) = cx + d \).

\[
\begin{align*}
f(g(x)) &= a(cx + d) + b = axc + ad + b \\
g(f(x)) &= c(ax + b) + d = axc + cb + d
\end{align*}
\]

5. Refer students to Calculator Note 4J.

Compositions are essentially a series of input-output functions. Drawing a vertical line up to the graph of \( g(x) \) gives the value of \( g(x) \). Drawing a horizontal line to the graph of \( y = x \) makes that \( y \)-value into an \( x \)-value. Drawing a vertical line to the graph of \( f(x) \) evaluates \( f(x) \) for that output value, and the horizontal line to the \( y \)-axis reveals the answer.

Assessing the Chapter

As a good resource for study, refer students to the table on page 232, Lesson 4.7, which includes a summary of all the transformations included in this chapter.

By the end of this chapter, students might be comfortable finding equations and graphing them without using their calculators. You might consider not using calculators on part of the chapter assessment.

FACILITATING SELF-ASSESSMENT

You might use some student-written items on the chapter assessment. Ask students to specify whether calculators will be allowed in solving the item they write.

Good portfolio items for this chapter include Lesson 4.1, Exercise 8; Lesson 4.2, Exercise 17; Lesson 4.3, Exercise 10; Lesson 4.4, Exercises 9 and 16; Lesson 4.5, Exercise 13; Lesson 4.6, Exercise 12; Lesson 4.7, Exercise 9; and Lesson 4.8, Exercise 9.

3. For the graph of the parent function \( y = x^2 \), you can think of any vertical dilation as an equivalent horizontal dilation. For example, the equations \( y = 4x^2 \) and \( y = (2x)^2 \) are equivalent, even though one represents a vertical dilation by a factor of 4 and the other represents a horizontal dilation by a factor of \( \frac{1}{2} \). For the graph of any function or relation, is it possible to think of any vertical dilation as an equivalent horizontal dilation? If so, explain your reasoning. If not, give examples of functions and relations for which it is not possible.

4. Enter two linear functions into \( f_1 \) and \( f_2 \) on your calculator. Enter the compositions of the functions as \( f_2(f_1(x)) \) and \( f_1(f_2(x)) \). Graph \( f_1 \) and \( f_2 \) and look for any relationships between them. (It will help if you turn off the graphs of \( f_1 \) and \( f_2 \)). Make a conjecture about how the compositions of any two linear functions are related. Change the linear functions in \( f_1 \) and \( f_2 \) to test your conjecture. Can you algebraically prove your conjecture?

5. One way to visualize a composition of functions is to use a web graph. Here’s how you evaluate \( f(g(x)) \) for any value of \( x \), using a web graph:

Choose an \( x \)-value. Draw a vertical line from the \( x \)-axis to the function \( g(x) \). Then draw a horizontal line from that point to the line \( y = x \). Next, draw a vertical line from this point so that it intersects \( f(x) \). Draw a horizontal line from the intersection point to the \( y \)-axis. The \( y \)-value at this point of intersection gives the value of \( f(g(x)) \).

Choose two functions \( f(x) \) and \( g(x) \). Use web graphs to find \( f(g(x)) \) for several values of \( x \). Why does this method work?