Chord Properties

In the last lesson you discovered some properties of a tangent, a line that intersects the circle only once. In this lesson you will investigate properties of a chord, a line segment whose endpoints lie on the circle.

In a person with correct vision, light rays from distant objects are focused to a point on the retina. If the eye represents a circle, then the path of the light from the lens to the retina represents a chord. The angle formed by two of these chords to the same point on the retina represents an inscribed angle.

First you will define two types of angles in a circle.

Investigation 1
Defining Angles in a Circle

Write a good definition of each boldfaced term. Discuss your definitions with others in your group. Agree on a common set of definitions as a class and add them to your definition list. In your notebook, draw and label a figure to illustrate each term.

Step 1 A central angle has its vertex at the center of the circle.

Central Angle.

\[ \angle AOB, \angle DOA, \text{ and } \angle DOB \]

are central angles of circle O.

\[ \angle PQR, \angle PQS, \angle RST, \angle QST, \text{ and } \angle QSR \]

are not central angles of circle P.

Step 2 An inscribed angle has its vertex on the circle and its sides are chords.

Inscribed Angle.

\[ \angle ABC, \angle BCD, \text{ and } \angle CDE \]

are inscribed angles.

\[ \angle PQR, \angle STU, \text{ and } \angle VWX \]

are not inscribed angles.

LESSON OBJECTIVES

- Discover properties of chords to a circle
- Practice construction skills

NCTM STANDARDS

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Step 3 How can you fold your circle construction to check the conjecture?

Step 4 Recall that the measure of an arc is defined as the measure of its central angle. If two central angles are congruent, their intercepted arcs must be congruent. Combine this fact with the Chord Central Angles Conjecture to complete the next conjecture.

"Pull the cord?! Don't I need to construct it first?"

Chord Arcs Conjecture
If two chords in a circle are congruent, then their intercepted arcs are congruent.
Investigation 3
Chords and the Center of the Circle

You will need
● a compass
● a straightedge
● patty paper (optional)

Step 1
Construct a large circle and mark the center. Construct two nonparallel congruent chords. Then construct the perpendiculars from the center to each chord.

Step 2
How does the perpendicular from the center of a circle to a chord divide the chord? Copy and complete the conjecture.

Perpendicular to a Chord Conjecture
The perpendicular from the center of a circle to a chord is the bisector of the chord.

Let’s continue this investigation to discover a relationship between the length of congruent chords and their distances from the center of the circle.

Step 3
Compare the distances (measured along the perpendicular) from the center to the chords. Are the results the same if you change the size of the circle and the length of the chords? State your observations as your next conjecture.

Chord Distance to Center Conjecture
Two congruent chords in a circle are equidistant from the center of the circle.

Investigation 4
Perpendicular Bisector of a Chord

You will need
● a compass
● a straightedge
● patty paper (optional)

Step 1
Construct a large circle and mark the center. Construct two nonparallel chords that are not diameters. Then construct the perpendicular bisector of each chord and extend the bisectors until they intersect.

Sharing Ideas (continued)
measure can have very different lengths. Ask whether having the same measure is enough to make two arcs congruent, as is the case for line segments and angles. [Ask] “What’s needed to ensure that two arcs have the same size and shape?” [Congruent arcs must be on the same or congruent circles.] (Some students may think that an arc of a larger circle will have the same size and shape as an arc of a smaller circle. To the extent possible, let other students convince them that the curvature will be different.)

[Ask] “Congruent chords are equidistant from the center. Can we say anything about distance from the center if one chord is longer than the other?” [In the same circle, shorter chords are farther from the center.]

Guiding Investigation 3
Step 1 Students can use the same two congruent chords and the same drawing for Investigations 2 and 3. [Alert] Students may need some review in how to construct perpendiculars. Using compass constructions to complete all four investigations may be too time-consuming, but this construction and the next can be completed quickly using patty-paper constructions. The perpendicular through the center of the circle to the chord can be folded.

Guiding Investigation 4
The perpendicular bisectors of the chords can be constructed by simply folding the chord in half. The Perpendicular Bisector of a Chord Conjecture is the converse of the Perpendicular to a Chord Conjecture.

Sharing Ideas
As usual, for presentations select groups that have a variety of statements of the conjectures. As students present, encourage them to put the ideas in their own words, not just those of the conjectures as presented in the student book. Help the class reach consensus on the wording to record in their notebooks.

[Ask] “How would you define congruent arcs?” Although the measure of an arc was first defined in Chapter 1, some students may still be wondering why arcs are measured in degrees based on their central angle. Drawing a central angle of 90° may help some students relate the central angle to a quarter of a circle. But one source of difficulty may be a natural tendency to think that measure means “size” in some direct way, and the measure of an arc doesn’t give its length. In fact, in different circles, arcs with the same
Assessing Progress
You can assess students’ understanding of (and use of the vocabulary for) radius, chord, central angle, and inscribed angle and their skill at constructing a circle, measuring an angle with a protractor, and comparing segments with a compass. You might also see how well they understand the difference between drawing and constructing.

Closing the Lesson
Summarize that the major conjectures of this lesson are about congruent chords of a circle: They determine congruent central angles, they intercept congruent arcs, and they are equidistant from the center. Another pair of conjectures about chords forms a biconditional: A line through the center of a circle is perpendicular to a chord if and only if it bisects the chord. If students are still shaky about these ideas, you might want to use one of Exercises 1–6 as an example.

BUILDING UNDERSTANDING
The exercises focus on applying the conjectures about chords. Encourage students to sketch pictures on their own papers. They can then mark and label all information accordingly.

ASSIGNING HOMEWORK
Essential 1–14, 17, 18
Performance assessment 17, 18
Portfolio 20
Journal 13, 14, 16
Group 15, 21
Review 22–28
Algebra review 15, 21, 26

MATERIALS
- circular objects and patty paper (Exercise 17)
- Exercises 18 and 19 (T), optional

EXERCISES
Solve Exercises 1–10. State which conjectures or definitions you used.

1. \( x = \frac{1}{2} \times 165° \)
2. \( z = \frac{1}{2} \times 84° \)
3. \( w = \frac{1}{2} \times 70° \)

4. **definition of measure of an arc**
   - \( AB = CD \)
   - \( PO = 8 \text{ cm} \)
   - \( OQ = \frac{1}{2} \times 8 \text{ cm} \)

5. **Chord Arcs Conjecture**
   - \( AB \) is a diameter. Find \( m\overset{\frown}{AC} \) and \( m\overset{\frown}{DB} \).

6. **Chord Central Angles Conjecture**
   - \( GIAN \) is a kite. Find \( w, x, \) and \( y \).
   - \( w = 115° \)
   - \( x = 115° \)
   - \( y = 65° \)

7. **Chord Distance to Center Conjecture**
   - \( AB = 6 \text{ cm} \)
   - \( OP = 4 \text{ cm} \)
   - \( CD = 8 \text{ cm} \)
   - \( OQ = 3 \text{ cm} \)
   - \( BD = 6 \text{ cm} \)
   - What is the perimeter of \( OPBQD \)? \( 20 \text{ cm} \)

8. **Perpendicular to a Chord Conjecture**
   - \( m\overset{\frown}{AC} = 130° \)
   - Find \( w, x, \) and \( y \).
   - \( w = 110° \)
   - \( x = 48° \)
   - \( y = 82° \)
   - \( z = 120° \)

9. **Chord Central Angles Conjecture**
   - \( x = \frac{1}{2} \times 96° \)
   - \( m\overset{\frown}{AC} = 68° \)
   - \( m\overset{\frown}{B} = 34° \) (Because \( \triangle OBC \) is isosceles, \( m\overset{\frown}{B} = m\overset{\frown}{C} \), \( m\overset{\frown}{B} + m\overset{\frown}{C} = 68° \), and therefore \( m\overset{\frown}{B} = 34° \).

Exercise 7 [Alert] Some students may neglect to add on the length \( OQ \).
Exercise 9 As needed, [Ask] “What do the measures of all the arcs add up to?” [360°]
10. $AB \parallel CO$, $m\overline{CO} = 66^\circ$
Find $x$, $y$, and $z$.

11. Developing Proof What's wrong with this picture?

12. Developing Proof What's wrong with this picture?

13. Draw a circle and two chords of unequal length. Which is closer to the center of the circle, the longer chord or the shorter chord? Explain.

14. Draw two circles with different radii. In each circle, draw a chord so that the chords have the same length. Draw the central angle determined by each chord. Which central angle is larger? Explain.

15. Polygon $MNOP$ is a rectangle inscribed in a circle centered at the origin. Find the coordinates of points $M$, $N$, and $O$.

$M(−4, 3), N(−4, −3), O(4, −3)$

16. Construction Construct a triangle. Using the sides of the triangle as chords, construct a circle passing through all three vertices. Explain. Why does this seem familiar?

17. Construction Trace a circle onto a blank sheet of paper without using your compass. Locate the center of the circle using a compass and straightedge. Trace another circle onto patty paper and find the center by folding.

18. Construction Adventurer Dakota Davis digs up a piece of a circular ceramic plate. Suppose he believes that some ancient plates with this particular design have a diameter of 15 cm. He wants to calculate the diameter of the original plate to see if the piece he found is part of such a plate. He has only this piece of the circular plate, shown at right, to make his calculations. Trace the outer edge of the plate onto a sheet of paper. Help him find the diameter.

19. Construction The satellite photo at right shows only a portion of a lunar crater. How can cartographers use the photo to find its center? Trace the crater and locate its center. Using the scale shown, find its radius. To learn more about satellite photos, go to www.keymath.com/DG. They can draw two chords and locate the intersection of their perpendicular bisectors. The radius is just over 5 km.

**Exercise 10** [Alert] Students may miss the fact that the two radii in $\triangle ABO$ are congruent, making it an isosceles triangle.

10. $x = 66^\circ$, $y = 48^\circ$, $z = 66^\circ$; Corresponding Angles Conjecture, Isosceles Triangle Conjecture, Linear Pair Conjecture

**Exercise 12** If students are having difficulty, [Ask] “What does the perpendicular bisector have to go through?” [the center of the circle]

13. The longer chord is closer to the center; the longest chord, which is the diameter, passes through the center.

14. The central angle of the smaller circle is larger, because the chord is closer to the center.

**Exercise 16** As needed, remind students of the meaning of a triangle inscribed in a circle (or a circle circumscribed around a triangle).

16. The center of the circle is the circumcenter of the triangle. Possible construction:

**Exercise 17** Have available round objects larger than coins for tracing.

17. possible construction:

**Exercises 17–19** If students are having difficulty, wonder aloud whether there's a conjecture that ends with something about the center of a circle. [Perpendicular Bisector of a Chord]

18. $\approx 13.8$ cm
20. **Developing Proof** Complete the flowchart proof shown, which proves that if two chords of a circle are congruent, then they determine two congruent central angles.

**Given:** Circle $O$ with chords $\overline{AB} \cong \overline{CD}$

**Show:** $\angle AOB \cong \angle COD$

**Flowchart Proof**

1. $\overline{AB} \cong \overline{CD}$
   - Given

2. $\overline{AO} \cong \overline{CO}$
   - All radii of a circle are congruent

3. $\overline{BO} \cong \overline{DO}$
   - All radii of a circle are congruent

4. $\triangle AOB \cong \triangle COD$
   - SSS Congruence Conjecture

5. $\angle AOB \cong \angle COD$
   - CPCTC

21. Circle $O$ has center $(0, 0)$ and passes through points $A(3, 4)$ and $B(4, -3)$. Find an equation to show that the perpendicular bisector of $\overline{AB}$ passes through the center of the circle. Explain your reasoning. $y = \frac{1}{2}x; (0, 0)$ is a point on this line.

**Review**

**Chapter 5**

22. **Developing Proof** Identify each of these statements as true or false. If the statement is true, explain why. If it is false, give a counterexample.

   a. If the diagonals of a quadrilateral are congruent, but only one is the perpendicular bisector of the other, then the quadrilateral is a kite. **true**

   b. If the quadrilateral has exactly one line of reflectional symmetry, then the quadrilateral is a kite. **false, isosceles trapezoid**

   c. If the diagonals of a quadrilateral are congruent and bisect each other, then it is a square. **false, rectangle**

23. **Mini-Investigation** Use what you learned in the last lesson about the angle formed by a tangent and a radius to find the missing arc measure or angle measure in each diagram. Examine these cases to find a relationship between the measure of the angle formed by two tangents to a circle, $\angle P$, and the measure of the intercepted arc, $\overline{AB}$. Then copy and complete the conjecture below.

   **Conjecture:** The measure of the angle formed by two intersecting tangents to a circle is $\angle (\text{Intersecting Tangents Conjecture})$. 180° minus the measure of the intercepted arc

22a.

Possible explanation: If $\overline{AB}$ is the perpendicular bisector of $\overline{CD}$, then every point on $\overline{AB}$ is equidistant from endpoints $C$ and $D$. Therefore $\overline{AC} \cong \overline{AD}$ and $\overline{BC} \cong \overline{BD}$. Because $\overline{CD}$ is not the perpendicular bisector of $\overline{AB}$, $C$ is not equidistant from $A$ and $B$. Likewise, $D$ is not equidistant from $A$ and $B$. So, $\overline{AC}$ and $\overline{BC}$ are not congruent, and $\overline{AD}$ and $\overline{BD}$ are not congruent. Thus $\overline{ACBD}$ has exactly two pairs of consecutive congruent sides, so it is a kite.