Factored Form

So far you have worked with quadratic equations in vertex form and general form. This lesson will introduce you to another form of quadratic equation, the factored form:

\[ y = a(x - r_1)(x - r_2) \]

This form helps you identify the roots, \( r_1 \) and \( r_2 \), of an equation. In the investigation you’ll discover connections between the equation in factored form and its graph. You’ll also use rectangle diagrams to convert the factored form to the general form and vice versa. Then in the example you’ll learn how to use a special property to find the roots of an equation.

Investigation

Getting to the Root of the Matter

First you’ll find the roots of an equation in factored form from its graph.

You will need

- graph paper

Step 1  
On your calculator, graph the equations \( y = x + 3 \) and \( y = x - 4 \) at the same time.

Step 2  
What is the \( x \)-intercept of each equation you graphed in Step 1?

Step 3  
Graph \( y = (x + 3)(x - 4) \) on the same set of axes as before. Describe the graph. Where are the \( x \)-intercepts of this graph?

Step 4  
Expand \( y = (x + 3)(x - 4) \) to general form. Graph the equation in general form on the same set of axes. What do you notice about this parabola and its \( x \)-intercepts? Is the graph of \( y = (x + 3)(x - 4) \) a parabola?
Now you'll learn how to find the roots from the general form.

Step 5  Complete the rectangle diagram whose sum is $x^2 + 5x + 6$. A few parts on the diagram have been labeled to get you started.

Step 6  Write the multiplication expression of the rectangle diagram in factored form. Use a graph or table to check that this form is equivalent to the original expression.

Step 7  Find the roots of the equation $0 = x^2 + 5x + 6$ from its factored form.

Step 8  Rewrite each equation in factored form by completing a rectangle diagram. Then find the roots of each. Check your work by making a graph.

a. $0 = x^2 - 7x + 10$

b. $0 = x^2 + 6x - 16$

c. $0 = x^2 + 2x - 48$

d. $0 = x^2 - 11x + 28$

Now you have learned three forms of a quadratic equation. You can enter each of these forms into your calculator to check that they are equivalent. Here are three equivalent equations that describe the height in meters, $y$, of an object in motion for $x$ seconds after being thrown upward. Each equation gives different information about the object.

**Vertex form**

$y = -4.9(x - 1.7)^2 + 15.876$

**General form**

$y = -4.9x^2 + 16.66x + 1.715$

**Factored form**

$y = -4.9(x + 0.1)(x - 3.5)$

Which form is best? The answer depends on what you want to know. The vertex form tells you the maximum height and when it occurs—in this case, 15.876 meters after 1.7 seconds (the vertex). The general form tells you that the object started at a height of 1.715 meters (the $y$-intercept). The coefficients of $x$ and $x^2$ give some information about the starting velocity and acceleration. The factored form tells you the times at which the object's height is zero (the roots).

You have already learned how to convert to and from the general form of a quadratic equation. Example A will show you how to get the vertex form from the factored form.

**EXAMPLE A**

Write the equation for this parabola in vertex form, factored form, and general form.

**Solution**

From the graph you can see that the $x$-intercepts are 3 and $-5$. So the factored form contains the binomial expressions $(x - 3)$ and $(x + 5)$.
If you graph \( y = (x - 3)(x + 5) \) on your calculator, you’ll see it has the same \( x \)-intercepts as the graph shown here, but a different vertex. The new vertex is \((-1, -16)\).

The new vertex needs to be closer to the \( x \)-axis, so you need to find the vertical shrink factor \( a \).

The original vertex of the graph shown is \((-1, -8)\). So the graph of the function must have a vertical shrink by a factor of \( \frac{-8}{-16} \), or 0.5.

The factored form is \( y = 0.5(x - 3)(x + 5) \). A calculator graph of this equation looks like the desired parabola.

Now you know that the value of \( a \) is 0.5 and that the vertex is \((-1, -8)\). Substitute this information into the vertex form to get \( y = 0.5(x + 1)^2 - 8 \).

Expand either form to find the general form.

<table>
<thead>
<tr>
<th>( x )</th>
<th>( x^2 )</th>
<th>( -3x )</th>
<th>( 5x )</th>
<th>(-15)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(-3)</td>
<td>(9)</td>
<td>(9)</td>
<td>(-45)</td>
<td>(45)</td>
</tr>
<tr>
<td>( x )</td>
<td>( x^2 )</td>
<td>( 1x )</td>
<td>( 1x )</td>
<td>(1)</td>
</tr>
<tr>
<td>( 1 )</td>
<td>(1)</td>
<td>(1)</td>
<td>(1)</td>
<td>(1)</td>
</tr>
</tbody>
</table>

So the three forms of the quadratic equation are

- **Vertex form** \( y = 0.5(x + 1)^2 - 8 \)
- **General form** \( y = 0.5x^2 + x - 7.5 \)
- **Factored form** \( y = 0.5(x - 3)(x + 5) \)
When finding roots it is helpful to use the factored form. In Example A, one root is 3 because 3 is the value that makes \((x - 3)\) equal to 0. The other root is 5 because it makes \((x + 5)\) equal to 0. Think of numbers that multiply to zero. If \(ab = 0\) or \(abc = 0\), the zero-product property tells you that \(a\), \(b\), or \(c\) must be zero. In an equation like \((x - 3)(x - 5) = 0\), at least one of the factors must be zero. The roots of an equation are sometimes called the zeros of a function because they make the value of the function equal to zero.

You can further explore the relationship between factored form, roots, and \(x\)-intercepts using the Dynamic Algebra Exploration at www.keymath.com/DA.

The ability to factor polynomials is also useful when simplifying rational expressions, as you'll see in the next example. When a polynomial equation is in factored form, and reduced, it's much easier to predict or identify characteristics such as \(x\)-intercepts and asymptotes.

**EXAMPLE B**

A rational expression can be reduced if there is a common factor in both the numerator and denominator. Reduce the expression \(\frac{x^2 + 2x - 24}{x^2 + 7x + 6}\) by factoring. Then check your answer with a graph.

**Solution**

First factor the quadratic expressions in the numerator and denominator. Using a rectangle diagram may help.

**Numerator:** \(x^2 + 2x - 24\)  
**Denominator:** \(x^2 + 7x + 6\)

So, \(x^2 + 2x - 24 = (x + 6)(x - 4)\) and \(x^2 + 7x + 6 = (x + 6)(x + 1)\). Now reduce the rational expression. Be sure to state any restrictions on the variable.

\[
\frac{x^2 + 2x - 24}{x^2 + 7x + 6} = \frac{(x + 6)(x - 4)}{(x + 6)(x + 1)} = \frac{(x + 6)(x - 4)}{(x + 6)(x + 1)} = \frac{x - 4}{x + 1}, \text{ where } x \neq -6 \text{ and } x \neq -1
\]

You can check your work by graphing \(y = \frac{x^2 + 2x - 24}{x^2 + 7x + 6}\) and \(y = \frac{x - 4}{x + 1}\). If the expressions are equivalent the graphs should be the same, except for any points that may be undefined in one graph, but defined in the other.
EXERCISES

Practice Your Skills

1. Use the zero-product property to solve each equation.
   a. \((x + 4)(x + 3.5) = 0\)
   b. \(2(x - 2)(x - 6) = 0\)
   c. \((x + 3)(x - 7)(x + 8) = 0\)
   d. \(x(x - 9)(x + 3) = 0\)

2. Graph each equation and then rewrite it in factored form.
   a. \(y = x^2 - 4x + 3\)
   b. \(y = x^2 + 5x - 24\)
   c. \(y = x^2 + 12x + 27\)
   d. \(y = x^2 - 7x - 30\)

3. Name the \(x\)-intercepts of the parabola described by each quadratic equation. Then check your answers with a graph.
   a. \(y = (x - 7)(x + 2)\)
   b. \(y = 2(x + 1)(x + 8)\)
   c. \(y = 3(x - 11)(x + 7)\)
   d. \(y = (0.4x + 2)(x - 9)\)

4. Write an equation of a quadratic function that corresponds to each pair of \(x\)-intercepts. Assume there is no vertical stretch or shrink.
   a. 2.5 and \(-1\)
   b. \(-4\) and \(-4\)
   c. \(-2\) and 2
   d. \(r_1\) and \(r_2\)

5. Consider the equation \(y = (x + 1)(x - 3)\).
   a. How many \(x\)-intercepts does the graph have?
   b. Find the vertex of this parabola.
   c. Write the equation in vertex form. Describe the transformations of the parent function, \(y = x^2\).

Reason and Apply

6. Is the expression on the left equivalent to the expression on the right? If not, change the right side to make it equivalent.
   a. \(x^2 + 7x + 12 \overset{?}{=} (x + 3)(x + 4)\)
   b. \(x^2 - 11x + 30 \overset{?}{=} (x + 6)(x + 5)\)
   c. \(2x^2 - 5x - 7 \overset{?}{=} (2x - 7)(x + 1)\)
   d. \(4x^2 + 8x + 4 \overset{?}{=} (x + 1)^2\)
   e. \(x^2 - 25 \overset{?}{=} (x + 5)(x - 5)\)
   f. \(x^2 - 36 \overset{?}{=} (x - 6)^2\)
7. Use a rectangle diagram to factor each expression.
   a. \(x^2 + 7x + 6\)  
   b. \(x^2 + 7x + 10\)  
   c. \(x^2 + x - 42\)  
   d. \(x^2 - 3x - 18\)  
   e. \(x^2 - 10x + 24\)  
   f. \(x^2 + 8x - 48\)

8. The sum and product of the roots of a quadratic equation are related to \(b\) and \(c\) in \(y = x^2 + bx + c\). The first row in the table below will help you to recognize this relationship.
   a. Complete the table.

<table>
<thead>
<tr>
<th>Factored form</th>
<th>Roots</th>
<th>Sum of roots</th>
<th>Product of roots</th>
<th>General form</th>
</tr>
</thead>
<tbody>
<tr>
<td>(y = (x + 3)(x - 4))</td>
<td>(-3) and 4</td>
<td>(-3 + 4 = 1)</td>
<td>((-3)(4) = -12)</td>
<td>(y = x^2 - 1x - 12)</td>
</tr>
<tr>
<td>(y = (x - 5)(x + 5))</td>
<td>5 and (-2)</td>
<td>(-5)</td>
<td>6</td>
<td></td>
</tr>
</tbody>
</table>

b. Use the values of \(b\) and \(c\) to find the roots of \(0 = x^2 + 2x - 8\).

9. **Mini-Investigation** In this exercise you will discover whether knowing the \(x\)-intercepts determines a unique quadratic equation. Work through the steps in 9a–e to find an answer. Graph each equation to check your work.
   a. Write an equation of a parabola with \(x\)-intercepts at \(x = 3\) and \(x = 7\).
   b. Name the vertex of the parabola in 9a.
   c. Modify your equation in 9a so that the graph is reflected across the \(x\)-axis. Where are the \(x\)-intercepts? Where is the vertex?
   d. Modify your equation in 9a to apply a vertical stretch with a factor of 2. Where are the \(x\)-intercepts? Where is the vertex? 
   e. How many quadratic equations do you think there are with \(x\)-intercepts at \(x = 3\) and \(x = 7\)? How are they related to one another?

10. Write a quadratic equation of a parabola with \(x\)-intercepts at \(-3\) and 9 and vertex at \((3, -9)\). Express your answer in factored form.

11. **APPLICATION** The school ecology club wants to fence in an area along the riverbank to protect endangered wildflowers that grow there. The club has enough money to buy 200 feet of fencing. It decides to enclose a rectangular space. The fence will form three sides of the rectangle, and the riverbank will form the fourth side.
   a. If the width of the enclosure is 30 feet, how much fencing material is available for the length? Sketch this situation. What is the area? 
   b. If the width is \(w\) feet, how much fencing material remains for the length, \(l\)?
   c. Use your answer from 11b to write an equation for the area of the rectangle in factored form. Check your equation with your width and area from 11a.
   d. Which two different widths would give an area equal to 0? 
   e. Which width will give the maximum area? What is that area?
12. **Mini-Investigation**  Consider the equation \( y = x^2 - 9 \).
   a. Graph the equation. What are the \( x \)-intercepts?
   b. Write the factored form of the equation.
   c. How are the \( x \)-intercepts related to the original equation?
   d. Write each equation in factored form. Verify each answer by graphing.
      i. \( y = x^2 - 49 \)
      ii. \( y = 16 - x^2 \)
      iii. \( y = x^2 - 47 \)
      iv. \( y = x^2 - 28 \)
   e. An expression in the form \( a^2 - b^2 \) is called a **difference of two squares**. Based on your work in 12a–d, make a conjecture about the factored form of \( a^2 - b^2 \).
   f. Graph the equation \( y = x^2 + 4 \). How many \( x \)-intercepts can you see?
   g. Explain the difficulty in trying to write the equation in 12f in factored form.

13. Kayleigh says that the roots of \( 0 = x^2 + 16 \) are 4 and \(-4 \) because \((4)^2 = 16\) and \((-4)^2 = 16\). Derek tells Kayleigh that there are no roots for this equation. Who is correct and why?

14. Reduce the rational expressions by dividing out common factors from the numerator and denominator. State any restrictions on the variable.
   a. \( \frac{(x - 2)(x + 2)}{(x + 2)(x + 3)} \)
   b. \( \frac{x^2 + 3x + 2}{(x - 4)(x + 2)} \)
   c. \( \frac{x^2 - 3x - 10}{x^2 - 5x} \)
   d. \( \frac{x^2 + 2x - 3}{x^2 + 5x + 6} \)
   e. \( \frac{x^2 + x - 6}{x^2 + 6x + 9} \)

**Review**

15. Multiply and combine like terms.
   a. \( (x - 21)(x + 2) \)
   b. \( (3x + 1)(x + 4) \)
   c. \( 2(2x - 3)(x + 2) \)

16. Edward is responsible for keeping the stockroom packed with the best-selling merchandise at the Super Store. He has collected data on sales of the new video game “Math-a-Magic.”

<table>
<thead>
<tr>
<th>Week</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>Games sold</td>
<td>0</td>
<td>186</td>
<td>366</td>
<td>516</td>
<td>636</td>
<td>727</td>
<td>789</td>
<td>821</td>
<td>825</td>
<td>798</td>
</tr>
</tbody>
</table>

[Data sets: GMWK, GMSLD]

   a. Find a quadratic model in vertex form that fits the data. Let \( w \) represent the week number and let \( s \) represent the number of games sold.
   b. If the pattern continues, in what week will people stop buying the game?
   c. How many total games will have been sold when people stop buying the game?
   d. There are 1000 games left in the stockroom at the start of week 11. How many more should Edward buy?